PROBLEM OF THE WEEK
Solution of Problem No. 14 (Fall 2001 Series)

Problem: Consider the motion of a mass in the (x, y) plane. Given that the angular velocity $\omega = x\dot{y} - y\dot{x}$ ($\cdot$ is $\frac{d}{dt}$) and the Lenz vector (ℓ_x, ℓ_y), where $\ell_x = -\frac{\omega}{k}\dot{y} + \frac{x}{r}$, $\ell_y = \frac{\omega}{k}\dot{x} + \frac{y}{r}$ $(r = (x^2 + y^2)^{1/2})$ are constant (independent of t), show that the acceleration vector points toward the point where $r = 0$ and its magnitude is inversely proportional to r^2.

Solution (by Mike Hamburg, Sr. St. Joseph H.S., South Bend, edited by the Panel)

First we note that $\dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r}$. Now, we have

$$\begin{aligned}
(\dot{\ell}_x, \dot{\ell}_y) &= \Big(-\frac{\omega}{k}\ddot{y} + \frac{\dot{x}r - \frac{x(x\dot{x}+y\dot{y})}{r}}{r^2}, \quad \frac{\omega}{k}\ddot{x} + \frac{\dot{y}r - \frac{x(x\dot{x}+y\dot{y})}{r}}{r^2}\Big) \\
&= \Big(-\frac{\omega}{k}\ \ddot{y} + \frac{\dot{x}y^2 - xy\dot{y}}{r^3}, \quad \frac{\omega}{k}\ddot{x} + \frac{x^2\dot{y} - xy\dot{x}}{r^3}\Big) \\
&= \Big(-\frac{\omega}{k}\ddot{y} - \frac{y\omega}{r^3}, \quad \frac{\omega}{k}\ddot{x} + \frac{x\omega}{r^3}\Big) \\
&= \omega\Big(-\frac{\ddot{y}}{k} - \frac{y}{r^3}, \quad \frac{\ddot{x}}{k} + \frac{x}{r^3}\Big).
\end{aligned}$$

If $\omega \neq 0$, since $(\dot{\ell}_x, \dot{\ell}_y) = 0$, it follows that the acceleration

$$(\ddot{x}, \ddot{y}) = -\frac{k}{r^2}\Big(\frac{x}{r}, \frac{y}{r}\Big),$$

which is parallel to (x, y), points toward the origin as long as $k > 0$, and has magnitude $\dfrac{|k|}{r^2}$ as required.

Mike Hamburg observes that the conclusion does not hold if $\omega = 0$.

Also solved by:

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