PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2004 Series)

Problem: Show that there are no rational numbers x, y such that

$$2x^2 - 3y^2 = 1,$$

but that there are infinitely many rational x, y such that

$$2x^2 - 3y^2 = -1.$$

Solution

Consider

$$2x^2 - 3y^2 = 1, \tag{1}$$

where x, y are rational. Then we get that there exists 3 integers p, q, r, without a common factor, such that

$$2p^2 - 3q^2 = r^2. \tag{2}$$

Any perfect square n^2 satisfies $n^2 \equiv 0$ or $n^2 \equiv 1 \bmod 3$. Since $2p^2 \equiv r^2 \bmod 3$, we see that p and r must be divisible by 3. Then q must be divisible by 3 as well, by (2). This contradicts the fact that p, q, r do not have a common factor. Therefore, (1) has no rational solution.

The following is a solution presented by Yuandong Tian (Sr., ECE), edited by the Panel.

Now, consider the second equation:

$$2x^2 - 3y^2 = -1. \tag{3}$$

One solution in rational numbers is (1,1). Let

$$y = 1 + t(x - 1), \qquad t \text{ rational} \tag{4}$$

be the line through (1,1) with rational slope t. It has at least one common point with the hyperbola (3), namely, (1,1). To find other intersection points, plug (4) into (3) to get the quadratic equation:

$$(2 - 3t^2)x^2 + (6t^2 - 6t)x + (-3t^2 + 6t - 2) = 0. \tag{5}$$

The exact form of this equation is not important; what is important is that it is quadratic equation with rational coefficients. Now, (5) has one rational root $x = 1$, therefore the other one is rational, too (the sum of the roots equals $(6t - 6t^2)/(2 - 3t^2)$). There are only two values of t for which those two roots are equal (and actually, they are irrational), so for all other rational t, (5) has a rational solution $x \neq 1$. Then (4) gives a corresponding rational y.

Explicit calculations, not necessary for the proof, show that

$$x = \frac{3t^2 - 6t + 2}{3t^2 - 2}, \quad y = \frac{-3t^2 + 4t - 2}{3t^2 - 2}$$

Solved by:

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