PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2005 Series)

Problem: Let $f(x, y, z)$ be a polynomial with real coefficients, of total degree ≤ 2, which takes on integer values at each of the 8 vertices of the unit cube $0 \leq x, y, z \leq 1$.

Show that f must take on odd values at an even number of the 8 vertices.

Solution (by the Panel)

Each such polynomial is a linear combination of the monomials

$$1, x, y, z, xy, yz, xz, x^2, y^2, z^2$$

with real (but not necessarily integral) coefficients.

We will show first that

$$\sum_{x,y,z\in\{0,1\}} (-1)^{x+y+z} P(x, y, z) = 0. \qquad (1)$$

Indeed, it is enough to prove (1) for each monomial that has the form $x^i y^j z^k$ with $i+j+k \leq 2$, i, j, k non–negative integers. In each such monomial at least one of the variables is missing, i.e., at least one of i, j, k equals 0. Let us say, for example that $k = 0$. Then we split the terms in (1), where $P = x^i y^j$, in two groups: one with $z = 0$, and the other one with $z = 1$. If we keep x, y fixed, the terms corresponding to $z = 0$ and $z = 1$ in (1) cancel each other. Therefore, all terms in (1) cancel.

Therefore, (1) is true for each monomial, thus it is true for P as well. Next, (1) implies easily that

$$\sum_{x,y,z\in\{0,1\}} P(x, y, z) \qquad \text{is even,}$$

and this yields the statement immediately.

Update on Problem # 2: It was also solved by Miguel Hurtado, grad. student, ECE.

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