PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2005 Series)

Problem: Assume that $a_n > 0$ for each n, and that

$$\sum_{n=1}^{\infty} a_n$$

converges. Prove that

$$\sum_{n=1}^{\infty} a_n^{\frac{n-1}{n}}$$

converges as well.

Solution I (by Georges Ghosn, Quebec)

We have for $n \geq 2$,

$$a_n^{\frac{n-1}{n}} = (a_n^{1/2} a_n^{1/2} \cdot a_n^{n-2})^{\frac{1}{n}} \leq \frac{2\sqrt{a_n} + (n-2)a_n}{n} \quad \text{(Arithmetic–geometric Inequality)}$$

But $\dfrac{2\sqrt{a}n}{n} \leq \dfrac{1}{n^2} + a_n$ (because $2xy \leq x^2 + y^2$),

and $\dfrac{(n-2)a_n}{n} \leq a_n$ (because $\dfrac{n-2}{n} \leq 1$).

Therefore, $0 < a_n^{\frac{n-1}{n}} \leq \dfrac{1}{n^2} + 2a_n$, for each $n \geq 1$. Finally the comparison test shows that $\displaystyle\sum_{n=1}^{\infty} a_n^{\frac{n-1}{n}}$ converges since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} + 2a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} + 2\sum_{n=1}^{\infty} a_n$ clearly converges.

Solution II (by the Panel)

Each term a_n satisfies either the inequality $0 < a_n \leq \dfrac{1}{2^n}$ or $\dfrac{1}{2^n} < a_n$. In the first case, $a_n^{\frac{n-1}{n}} \leq \dfrac{1}{2^{n-1}}$. In the second one, $a_n^{\frac{n-1}{n}} = \dfrac{a_n}{a_n^{\frac{1}{n}}} \leq 2a_n$.

Therefore, in both cases,

$$0 < a_n^{\frac{n-1}{n}} \leq \frac{1}{2^n} + 2a_n.$$

The conclusion is now immediate since $\sum \frac{1}{2^n}$ converges, and so does $\sum 2a_n$.

There were no other correct solutions to this problem.