PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2005 Series)

Problem: Triangle $T_1 = \triangle A_1B_1C_1$ is included in circle K. The perpendicular bisectors are drawn and extended through the interior of T_1 to their intersections A_2, B_2, C_2 with K. This process is repeated with the new triangle $T_2 = \triangle A_2B_2C_2$ to get new points A_3, B_3, C_3, etc.

Prove that

(a) the sequence T_n has a subsequence that converges to some triangle T_∞ and

(b) T_∞ must be equilateral.

Solution (by the Panel)

First, (a) holds for any sequence of inscribed triangles $T_n = \triangle A_nB_nC_n$ by the following argument. Since A_n belong to a compact set (the circle K), there is a convergent subsequence $A_{n_k} \to A_\infty \in K$. Apply the same argument to B_{n_k} to get a convergent sub–subsequence $B_{n_{k_j}} \to B_\infty \in K$. Then, of course, $A_{n_{k_j}} \to A_\infty$. Finally, repeat this argument one more time to get a subsequence $C_{n_{k_{j_i}}} \to C_\infty \in K$. Then $T_{n_{k_{j_i}}}$ converges to $T_\infty = A_\infty B_\infty C_\infty$.

We need to show that in our case, any such T_∞ will be equilateral. Let $\alpha_n, \beta_n, \gamma_n$, be the angles of T_n. It is easy to show that

$$\alpha_{n+1} = \frac{\pi}{2} - \frac{\alpha_n}{2}, \quad \beta_{n+1} = \frac{\pi}{2} - \frac{\beta_n}{2}, \quad \gamma_{n+1} = \frac{\pi}{2} - \frac{\gamma_n}{2}.$$

Those are recurrence equations with solutions

$$\alpha_n = \frac{-2\alpha_1}{(-2)^n} + \frac{2\pi}{3(-2)^n} + \frac{\pi}{3},$$

similarly for β_n, γ_n. Therefore, $\alpha_n \to \pi/3$, $\beta_n \to \pi/3$, $\gamma_n \to \pi/3$. Any subsequence has the same limit. Therefore, T_∞ must be equilateral.

As George Ghosn pointed out, actually the whole sequence T_n converges (to an equilateral triangle).

At least partially solved by:

<u>Graduates</u>: Eu Jin Toh (ECE)

<u>Others</u>: Stephen Casey (Ireland), Georges Ghosn (Quebec), Wing–Kai Hon (Post-doc, CS), Steven Landy (IUPUI Physics staff)