PROBLEM OF THE WEEK
Solution of Problem No. 4 (Fall 2006 Series)

Problem: Given that (x_0, y_0), $y_0 \neq 0$ is a rational point on the curve $y^2 = x^3+ax^2+bx+c$, with a, b, c rational and that (x_0, y_0) is not an inflection point, find two more rational points on the curve.

Solution (by Jonathan Landy, UCLA, edited by the Panel)

If (x_0, y_0) is a rational point on the curve, then so is the point $(x_0, -y_0)$ (this is a distinct point as $y_0 \neq 0$). To find a third rational point on the curve, we will find the intersection of the tangent line to the curve at (x_0, y_0) with the curve. The equation for this tangent line is

$$\frac{y - y_0}{x - x_0} = \frac{3x_0^2 + 2ax_0 + b}{2y_0}.$$

A second intersection point of this line with the curve may be found by setting the respective y^2 values equal, giving

$$\left[(x - x_0) \cdot \frac{3x_0^2 + 2ax_0 + b}{2y_0} + y_0\right]^2 = x^3 + ax^2 + bx + c.$$

Rearrangement gives,

$$(x - x_0)^2 - \left\{x + a + 2x_0 - \left(\frac{3x_0^2 + 2ax_0 + b}{2y_0}\right)^2\right\} = 0.$$

A second intersection point of the line with the curve is thus given by (x_3, y_3), where

$$x_3 = \left(\frac{3x_0^2 + 2ax_0 + b}{2y_0}\right) - a - 2x_0,$$

and

$$y_3 = (x_3 - x_0)\left(\frac{3x_0^2 + 2ax_0 + b}{2y_0}\right) + y_0.$$

This point is rational and distinct from both (x_0, y_0) and $(x_0, -y_0)$, because (x_0, y_0) is not an inflection point.
This is a third rational point on the curve.

At least partially solved by:

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