PROBLEM OF THE WEEK
Solution of Problem No. 6 (Fall 2006 Series)

Problem:

Show that there exists a constant C such that for any sequence $\{a_n\}$ with positive terms,

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + \cdots + a_n} \le C \sum_{n=1}^{\infty} \frac{1}{a_n}$$

whenever the series on the right–hand side converges.

Hint: Consider first monotone sequences $\{a_n\}$.

Solution (by Georges Ghosn, Quebec, edited by the Panel)
We consider first an increasing sequence $\{a_n\}$ with positive terms. For any given $n \ge 1$, we have:

$$\frac{2n}{a_1 + a_2 + \cdots + a_{2n}} \le \frac{2n}{a_{n+1} + \cdots + a_{2n}} \le \frac{2n}{na_n} = \frac{2}{a_n}$$

and

$$\frac{2n+1}{a_1 + \cdots + a_{2n+1}} \le \frac{2n+1}{a_{n+1} + \cdots + a_{2n+1}} \le \frac{2n+1}{(n+1)a_n} \le \frac{2}{a_n}.$$

Therefore:

$$\sum_{n=1}^{N} \frac{n}{a_1 + \cdots + a_n} \le \frac{1}{a_1} + \frac{2}{a_1} + \frac{2}{a_1} + \frac{2}{a_2} + \frac{2}{a_2} + \cdots + \frac{2}{a_{[\frac{N}{2}]}} \le 5 \sum_{n=1}^{[\frac{N}{2}]} \frac{1}{a_n}$$

where $\left[\frac{N}{2}\right]$ is the integer part of $\frac{N}{2}$.
Hence, by the comparison test we deduce:

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + \cdots + a_n} \le 5 \sum_{n=1}^{\infty} \frac{1}{a_n}.$$

Consider now a sequence $\{a_n\}$ with positive terms. Since reordering the terms does not affect the value towards which the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, we can define an increasing sequence $\{b_n\}$ by reordering the terms of the sequence $\{a_n\}$. Since $a_n \to \infty$, it is easy to see that such reordering exists. Therefore $\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{b_n}$. But from above we have $\sum_{n=1}^{\infty} \frac{n}{b_1 + \cdots + b_n} \le 5 \sum_{n=1}^{\infty} \frac{1}{b_n}$, and $\forall n \ge 1$, $\frac{n}{a_1 + \cdots + a_n} \le \frac{n}{b_1 + \cdots + b_n}$, because

$b_1 \dots b_n$ are the first smallest n terms of the sequence $\{a_n\}$. Therefore, the comparison test gives:

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + \cdots + a_n} \leq \sum_{n=1}^{\infty} \frac{n}{b_1 + \cdots + b_n} \leq 5 \sum_{n=1}^{\infty} \frac{1}{b_n} = 5 \sum_{n=1}^{\infty} \frac{1}{a_n}$$

Finally C exists and $C = 5$ is one of its possible values.

We can show that C can be chosen less or equal to e.

Indeed, $\dfrac{n}{a_1 + \cdots + a_n} \leq \dfrac{1}{(a_1 \times \cdots \times a_n)^{\frac{1}{n}}} = \left(\dfrac{1}{a_1} \times \cdots \times \dfrac{1}{a_n}\right)^{\frac{1}{n}}$ $(A - G$ Inequality) and

$\displaystyle\sum_{n=1}^{N} \left(\frac{1}{a_1} \times \cdots \times \frac{1}{a_n}\right)^{\frac{1}{n}} \leq e \sum_{n=1}^{N} \frac{1}{a_n}$ (Carleman's Inequality)

Therefore, the inequality is true with $C = e$.

——There are no other correct solutions for this problem.——