PROBLEM OF THE WEEK

Solution of Problem No. 7 (Fall 2006 Series)

**Problem:**

Given a triangle $\triangle$, let $d(P)$, $e(P)$, $f(P)$ denote the distances of a point $P$ inside $\triangle$ from the three sides of $\triangle$ and let

$$M(P) = \max(d(P), e(P), f(P)).$$

Prove that $Q$ in $\triangle$ is the center of the inscribed circle of $\triangle$ if and only if

$$M(Q) < M(P) \quad \text{for all} \quad P \neq Q,\ P \text{ in } \triangle.$$

**Solution** (by the Panel)

Let $a, b, c$ be the sides of the triangle $\triangle$. Then

$$ad(P) + be(P) + cf(P) = 2A, \tag{1}$$

where $A$ is the area of $\triangle$. If $P = Q$, then

$$r(a+b+c) = 2A,$$

where $r = d(Q) = e(Q) = f(Q) = M(Q)$. Then (1) yields

$$(a+b+c)\, M(P) \ \geq\ 2A \ =\ (a+b+c)\, M(Q)$$

with equality if and only if $P = Q$. This completes the proof.

At least partially solved by:

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