PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2008 Series)

Problem: Show that $\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^2 dx_1,\ldots,d_n = \frac{1}{4}$.

Solution (by Manuel Barbéro, New York)

Let's define for all $1 \le i \ne j,\ k \le n$:

$$P_k = \int_0^1 \cdots \int_0^1 x_k^2 dx_1 \ldots dx_n = \left(\int_0^1 dx_1\right) \ldots \left(\int_0^1 x_k^2 dx_k\right) \ldots \left(\int_0^1 dx_n\right)$$
$$= 1 \ldots \left(\frac{x_k^3}{3}\bigg|_0^1\right) \cdots = \frac{1}{3}$$

$$Q_{i,j} = \int_0^1 x_i x_j dx_1 \ldots dx_n = \left(\int_0^1 dx_1\right) \ldots \left(\int_0^1 x_i dx_i\right) \ldots \left(\int_0^1 x_j dx_j\right) \ldots \left(\int_0^1 dx_n\right)$$
$$= \left(\frac{x_i^2}{2}\bigg|_0^1\right) \cdots \left(\frac{x_j^2}{2}\bigg|_0^1\right) = \frac{1}{4}$$

In fact, P_k, $Q_{i,j}$ are independent of i, j or k and since $(x_1 + \cdots + \cdots + x_n)^2 = \sum_{k=1}^{n} x_k^2 + \sum_{1\le i\ne j\le n}^{n} x_i x_j$, we have

$$I_n = \frac{1}{n^2}\left(\sum_{k=1}^{k=n} P_k + \sum_{1\le i\ne j\le n}^{k=n} Q_{i,j}\right) = \frac{1}{n^2}\left(n\frac{1}{3} + n(n-1)\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{12n} \overset{n\to\infty}{\longrightarrow} \frac{1}{4}.$$

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