PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2008 Series)

Problem: Show that if m, n are positive integers then the smaller of $\sqrt[n]{m}$ and $\sqrt[m]{n}$ is no larger than $\sqrt[3]{3}$.

Solution (by Huanyu Shao, Graduate student, Computer Science, Purdue University)

Assume $m \le n$. Then $\frac{1}{m} \ge \frac{1}{n} > 0$ then $m^{\frac{1}{n}} \le m^{\frac{1}{m}}$ (because m is a positive integer). So, the smaller of $m^{\frac{1}{n}}$, $n^{\frac{1}{m}}$ is no larger than the larger of $m^{\frac{1}{m}}$ and $n^{\frac{1}{n}}$.

We then try to prove that $\max\limits_{m \in N} m^{\frac{1}{m}} = \sqrt[3]{3}$.

Let

$$f(x) = x^{\frac{1}{x}} \ (x > 0)$$

$$f'(x) = (e^{\frac{\ln x}{x}})' = e^{\frac{\ln x}{x}} \cdot \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = x^{\frac{1}{x}} \cdot \frac{1 - \ln x}{x^2}.$$

So $f'(x) > 0$ when $x < e$, $f'(x) < 0$ when $x > e$. So f decreases when $x > e$.

$m^{\frac{1}{m}}$ decreases when $n \ge 3$. And we also have $\sqrt[1]{1} < \sqrt[2]{2} < \sqrt[3]{3}$.

So $\max\limits_{m \in N} m^{\frac{1}{m}} = \sqrt[3]{3}$.

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