PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2008 Series)

Problem: A piece is broken off at random from each of three identical rods. What is the probability that an acute triangle can be formed from the three pieces?

Solution (by Michael Burkhart, Sophomore, Econ. Purdue Univ.)

Let three rods of length L be broken into rods of lengths: x, y, z. If each measurement is assigned a separate axis, the set of all possible outcomes (x, y, z) is distributed evenly over $(0, L) \times (0, L) \times (0, L)$ which has a volume of L^3. The subset of outcomes which combine to form acute triangles is characterized by the following criteria:

$$\begin{cases} x^2 + y^2 > z^2 \\ x^2 + z^2 > y^2 \\ y^2 + z^2 > x^2 \end{cases}$$

Nota Bene: Here the triangle inequality is a superfluous restraint because: $\forall (x, y, z > 0)$:

$$x^2 + y^2 > z^2 \Rightarrow x^2 + 2xy + y^2 > z^2 \Leftrightarrow (x + y)^2 > z^2 \Leftrightarrow x + y > z.$$

The volume enclosed by the solution subset is:

$$\begin{aligned} & L^3 - \int_0^L \left(\frac{\pi x^2}{4}\right) dx - \int_0^L \left(\frac{\pi y^2}{4}\right) dy - \int_0^L \left(\frac{\pi z^2}{4}\right) dz \\ &= L^3 - \left.\frac{\pi x^3}{12}\right|_0^L - \left.\frac{\pi y^3}{12}\right|_0^L - \left.\frac{\pi z^3}{12}\right|_0^L \\ &= L^3 \left(1 - \frac{\pi}{4}\right). \end{aligned}$$

The probability that the random lengths x, y, z constrained by $(0, L)$ will fall in the solution space is :

$$\frac{L^3 \left(1 - \frac{\pi}{4}\right)}{L^3} = 1 - \frac{\pi}{4}.$$

Also solved by:

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