PROBLEM OF THE WEEK
Solution of Problem No. 12 (Fall 2009 Series)

Problem: Find, with proof, the maximum value of $\prod_{j=1}^{k} x_j$ where $x_j \geq 0$, $\sum_{j=1}^{k} x_j = 100$, and k is variable. In particular, your answer must be greater than or equal to all values obtained from other choices of k.

Solution (by Craig Schroeder, PhD. student, Stanford Univ.)

Assume that $k > 1$ and that, without loss of generality, $x_1 \neq x_2$. Let $a = b = \frac{x_1 + x_2}{2}$, $a + b = x_1 + x_2$, but $ab - x_1 x_2 = \left(\frac{x_1 - x_2}{2}\right)^2 > 0$. Thus, the optimum value must be obtained when $x_1 = x_2 = \cdots = x_k$. Let this value be x. Since $kx = 100$, $x = \frac{100}{k}$. The value to be maximized is $x^k = \left(\frac{100}{k}\right)^k$. Maximizing $\left(\frac{100}{k}\right)^k$ is the same as maximizing

$$f(k) = \ln\left(\frac{100}{k}\right)^k = k \ln 100 - k \ln k.$$
$$f'(k) = \ln 100 - \ln k - 1.$$
$$f''(k) = -\frac{1}{k} < 0.$$

$$0 = f'(k) = \ln 100 - \ln k - 1 \;\Leftrightarrow\; k = 100e^{-1} \approx 36.8.$$

The first derivate tells us the optimum (if k could be any real), and the second derivative tells us that this is a maximum. The first derivative also tells us that $f(k)$ increases if k is less than this and decreases if greater. Since k must be an integer, the two candidates for the maximum are $k = 36$ and $k = 37$. $f(36) \approx 36.779$ and $f(37) \approx 36.787$. Thus, the solution is $k = 37$. The product is $\left(\frac{100}{37}\right)^{37} \approx 9.4741 \times 10^{15}$.

The problem was also completely or partially proved by:

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