PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2009 Series)

Problem: Let C be a closed convex curve with continuously turning tangent. Prove that, if $\triangle$ is an inscribed triangle of maximal perimeter, then the normals to C at the vertices of $\triangle$ bisect the angles of $\triangle$.

Solution (by Andrea Altamura, Graduate student, Italy)

Let P_0, P_1 and P_2 the vertices of the triangle of maximal perimeter inscribed in C. Without loss of generality we can assume that $P_0 = (0,0)$ and $P_1 = (a,0)$. Let

$$C = \{(x(t), y(t)) : t \in [0,1]\}, \quad P_t = (x(t), y(t)).$$

Consider $p(t)$ the perimeter of the $\triangle$ determined by P_0, P_1 and P_t, that is

$$p(t) = |P_t - P_0| + |P_t - P_1| + |P_1 - P_0| = \sqrt{x(t)^2 + y(t)^2} + \sqrt{(x(t)-a)^2 + y(t)^2} + |a|$$

for $t \in [0,1]$. Finding the critical points, we get

$$\begin{aligned} 0 &= p'(t) \\ &= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}} + \frac{(x(t)-a)x'(t) + y(t)y'(t)}{\sqrt{(x(t)-a)^2 + y(t)^2}} \\ &= \frac{\langle x(t), y(t)\rangle \cdot \langle x'(t), y'(t)\rangle}{|P_t - P_0|} + \frac{\langle x(t)-a, y(t)\rangle \cdot \langle x'(t), y'(t)\rangle}{|P_t - P_1|} \\ &= |\langle x'(t), y'(t)\rangle| \cos(\theta_0) - |\langle x'(t), y'(t)\rangle| \cos(\theta_1) \end{aligned}$$

where θ_0 is the angle between the vector $P_t - P_0$ and the tangent line of C at P_t, and θ_1 is the angle between the $P_t - P_1$ and the tangent line of C at P_t (with opposite direction). Since the curve has continuously turning tangent we can choose the parametrization such that $|\langle x'(t), y'(t)\rangle| \neq 0$. Thus $\theta_0 = \theta_1$. Now, since C is convex, this is the same that saying that at all critical points of $p(t)$ the normal of C at P_t bisects the angles of the $\triangle(P_0, P_1, P_t)$ at P_t. Since P_2 is a critical point of $p(t)$ the proof is finished.

The problem was also solved by:

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