PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2010 Series)

Problem: A particle moves in three–space according to the equations

$$\frac{dx}{dt} = yz, \quad \frac{dy}{dt} = xz, \quad \frac{dz}{dt} = xy.$$

Show that

(a) if two of $x(0), y(0), z(0)$ are zero, the particle never moves;

(b) otherwise, either the particle moves to infinity at some finite time in the future or it came from infinity at some finite time in the past.

**Note for the non–expert reader:

All of the correct solutions tacitly use the standard (local) existence and uniqueness theorem. For this system of equations it implies that the maximally defined solution, for initial conditions at $t = 0$, is uniquely determined by the initial conditions and is defined on some open interval (t_-, t_+). If $t_+ < \infty$, then the particle moves to ∞ at time t_+; and if $t_- > -\infty$, then the particle came from ∞ at time t_-.

Solution: (by Denes Molnar, Faculty, Physics Department)

First notice that $x^2 - y^2, y^2 - x^2$ and $x^2 - z^2$ are constants of motion, i.e., $y^2(t) = x^2(t) + a, z^2(t) = x^2(t) + b$. Due to invariance under joint flipping of any two signs (e.g., $x(t) \to -x(t), y(t) \to -y(t)$), there are without loss of generality two classes of initial conditions with x, y, z all non–zero:

i) $x_0 > 0, y_0 > 0, z_0 > 0$ (at $t = t_0$):
In this case x, y, z grow monotonically and stay positive for all $t > t_0$, i.e., $\dot{x} = \sqrt{x^2 + a}\sqrt{x^2 + b}$ and

$$dt = \frac{dx}{\sqrt{x^2 + a}\sqrt{x^2 + b}} \qquad (1)$$

$$t_\infty - t_0 = \int_{x_0}^{\infty} \frac{dx}{\sqrt{x^2 + a}\sqrt{x^2 + b}} = finite > 0 \qquad (2)$$

because asymptotically the integrand is $\sim 1/x^2$. Hence the particle goes to ∞ at the finite time t_∞.

ii) $x_0 > 0, y_0 > 0, z_0 < 0$ (at $t = t_0$):
In this case x, y increase monotonically and stay positive, while z decreases monotonically and stays negative, as we evolve backwards for all $t < t_0$. I.e., $\dot{x} = \sqrt{x^2+a}(-\sqrt{x^2+b})$ and

$$t_\infty - t_0 = \int_{x_0}^{\infty} \frac{dx}{-\sqrt{x^2+a}\sqrt{x^2+b}} = finite < 0 \tag{3}$$

for the same reason (asymptotics).

If at least two of the variables are zero at $t = 0$, then all derivatives are zero and x, y, z maintain their initial values at all times (including $t < 0$).

If precisely one variable, say x, is zero initially, then via sign flipping we can ensure $y > 0$, $z > 0$, i.e., $\dot{x} > 0$. Hence for small enough $\epsilon > 0$, at $t = \epsilon$ all three variables will be positive, and case i) applies with $t_0 = \epsilon$.

Also completely or partially solved by:

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