PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2010 Series)

Problem: For each positive integer n, let t_n denote the number of divisors (including 1 and n) of n.

Prove that

$$t_1 + \cdots + t_n = \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \cdots + \left[\frac{n}{n}\right].$$

Note: If x is any real number, then $[x]$ denotes the greatest integer m satisfying $m \leq x$.

Solution (by Dong–Gil Shin, 12th grade, Newton North High School)

$[n/x]$ for some $x \leq n$ just represents the number of natural numbers less than or equal to n that are divisible by x. Thus if we have a list of all divisors of $1, 2, 3, \ldots$ and n (of course repetitions included), 1 will occur $[n/1]$ times, 2 will occur $[n/2]$ times, and so on. Since $1, \ldots, n$ are the only possible divisors for numbers $1, \ldots, n$, $[n/1] + [n/2] + \cdots + [n/n]$ includes all the terms in the list, thus $t_1 + t_2 + \cdots + t_n = [n/1] + [n/2] + \cdots + [n/n]$.

The problem was also solved by:

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