PROBLEM OF THE WEEK
Solution of Problem No. 1 (Fall 2012 Series)

Problem: Given a collection of five non-zero vectors in three space show that two must have a non-negative inner product.

Solution: (by Kilian Cooley, Senior, Math & AAE, Purdue University)

Denote the five vectors by $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_5}$. Begin by showing that $\vec{v_i} \neq c\vec{v_j}$ for any indices $i \neq j$ and nonzero real c. If $c > 0$ and $\vec{v_i} = c\vec{v_j}$ for some i and j, then $\vec{v_i} \cdot \vec{v_i} = c\vec{v_i} \cdot \vec{v_j} < 0$, however $\vec{v_i}$ must have positive length so this is a contradiction. Suppose $c < 0$ and let k be an index different from i and j. Then $\vec{v_i} \cdot \vec{v_k} = c\vec{v_j} \cdot \vec{v_k}$. But since all the inner products as well as c are strictly negative, the two sides of this equation have opposite signs. Therefore $\vec{v_i} \neq c\vec{v_j}$ for all indices i and j and all nonzero c.

Since the signs of the inner products remain unchanged if the vectors are scaled by positive constants, it may be assumed that all the $\vec{v_j}$ have unit length. Consider the orthogonal projections $\vec{w_2}, \vec{w_3}, \vec{w_4}, \vec{w_5}$ of $\vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}$ respectively onto the plane normal to $\vec{v_1}$. Then for $2 \leq k \leq 5$,

$$\vec{w_k} = \vec{v_k} - (\vec{v_k} \cdot \vec{v_1})\vec{v_1}$$

The foregoing lemma implies that all the $\vec{w_k}$ are nonzero. Consider the inner product

$$\begin{aligned}
\vec{w_p} \cdot \vec{w_q} &= (\vec{v_p} - (\vec{v_p} \cdot \vec{v_1})\vec{v_1} \cdot (\vec{v_q} - (\vec{v_q} \cdot \vec{v_1})\vec{v_1} \\
&= \vec{v_p} \cdot \vec{v_q} - (\vec{v_p} \cdot \vec{v_1})(\vec{v_1} \cdot \vec{v_q}) - (\vec{v_q} \cdot \vec{v_1})(\vec{v_1} \cdot \vec{v_p}) + (\vec{v_p} \cdot \vec{v_1})(\vec{v_q} \cdot \vec{v_1})(\vec{v_1} \cdot \vec{v_1}) \\
&= \vec{v_p} \cdot \vec{v_q} - (\vec{v_p} \cdot \vec{v_1})(\vec{v_1} \cdot \vec{v_q})(2 - \vec{v_1} \cdot \vec{v_1}) \\
&= \vec{v_p} \cdot \vec{v_q} - (\vec{v_p} \cdot \vec{v_1})(\vec{v_1} \cdot \vec{v_q}) < 0
\end{aligned}$$

Since $\vec{v_1}$ has unit length and all inner products among the are negative. This, however, implies that the angle between any two $\vec{w_k}$ exceeds $\pi/2$ which is impossible since the four angles formed by the vectors must sum to 2π. This contradiction implies that at least two of the five $\vec{v_j}$ have a non-negative inner product.

The problem was also solved by:

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