

PROBLEM OF THE WEEK  
Solution of Problem No. 1 (Fall 2013 Series)

**Problem:**

Let  $a_n > 0$ ,  $n \geq 0$ . Call  $k$  “good” if  $k \geq 1$  and  $a_k > \frac{1}{2}a_{k-1}$ . Also call 0 good.

Show  $\sum_{k=0}^{\infty} a_k$  converges if  $\sum_{\text{good } k} a_k$  converges.

**Solution:** (by David Stoner)

For each nonnegative integer  $k$ , define  $f(k)$  to be the least positive integer greater than  $k$  which is good. If no such integers exist, take  $f(k) = \infty$ . Define  $S_k = \sum_{i=k}^{f(k)-1} a_i$  (this is  $\sum_{i=k}^{\infty} a_i$  if  $f(k) = \infty$ .)

Lemma: When  $k$  is good,  $S_k \leq 2a_k$ .

*Proof:* If  $f(k) = k + 1$ , then  $S_k = a_k$  and the result is clearly true. Otherwise, for each good  $k$ , by definition all integers  $n$  with  $k < n < f(k)$  satisfy  $a_n \leq \frac{a_{n-1}}{2}$ . By repeated applications of this, we have:

$$a_{k+i} \leq \frac{a_k}{2^i}$$

for integers  $0 < i < f(k) - k$ . This means that  $S_k = a_k + \sum_{i=1}^{f(k)-k-1} a_{k+i} \leq a_k + \sum_{i=1}^{\infty} \frac{a_k}{2^i} = 2a_k$

as desired. So the lemma is proved. Now note that since  $a_0$  is good, we have  $\sum_{k=0}^{\infty} a_k =$

$\sum_{\text{good } k} S_k$ . But  $0 < \sum_{\text{good } k} S_k \leq \sum_{\text{good } k} 2a_k$  by the lemma, and the latter converges by assumption.

Therefore,  $\sum_{k=0}^{\infty} a_k$  converges as desired.

**The problem was also solved by:**

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