PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2014 Series)

Problem:

Form a "triangle" with 10 blocks in its top row, 9 blocks in the next row, etc., until the bottom row has one block. Each row is centered below the row above it. Color the blocks in the top row red, white and blue in any way. Use these two rules to color the remaining rows of the triangle:

- **If two consecutive blocks in a row have the same color, the block between them in the row below has the same color.**
- **If two consecutive blocks in a row have different colors, the block between them in the row below has the third color.**

Tell how you can always predict the color of the bottom block after seeing only the top row (and not constructing the intermediate rows). Prove your answer.

<u>**Solution by the Committee**</u>

Rename the colors 0, 1, 2. Let B denote a block, which may have any one of the three colors. Then B^n denotes a row of n blocks.

Define a function $f : B \times B \to B$ that gives the color of a block from the colors of the two blocks above it, using the rules above. Thus if a and b are colors (integers modulo 3), then $f(a, a) = a$ and $f(a, b) = c$ if a, b, c are 0, 1, 2, in some order. It is easy to check that $f(a, b) = (-a - b) \bmod 3$ in all nine cases. (Or note that $-2a \equiv a \pmod 3$ and $0 + 1 + 2 \equiv 0 \pmod 3$.)

Now f induces a function $f_n : B^n \to B^{n-1}$ that computes the colors in the next row from those in the row above. In fact,

$$f_n(a_1, a_2, \dots, a_n) = (f(a_1, a_2), f(a_2, a_3), f(a_3, a_4), \dots f(a_{n-1}, a_n)).$$

The $n-1$-fold composition $F_n = f_2 \circ f_3 \circ \cdots \circ f_n : B^n \to B$ computes the color of the bottom block from the colors in the n-th row.

We will prove that $F_{10}(a_1, a_2, \dots, a_{10}) = f(a_1, a_{10})$ is independent of a_2, a_3, …, a_9.

Using the property $f(a, b) = -(a + b) \bmod 3$, one proves by induction on n that

$$F_n(a_1, a_2, \dots, a_n) \equiv (-1)^{n-1} \left(a_1 + \binom{n-1}{1} a_2 + \cdots + \binom{n-1}{n-2} a_{n-1} + a_n \right) \pmod 3.$$

In our case, $n = 10$ and the ninth row of Pascal's triangle is

$$1\ 9\ 36\ 84\ 126\ 126\ 84\ 36\ 9\ 1.$$

Each number $\binom{9}{i}$ in the row is divisible by 3, except for the two 1s. Therefore, $F_{10}(a_1, a_2, \ldots, a_{10}) \equiv -a_1 - a_{10} = f(a_1, a_{10})$. This shows that the color of the bottom block is determined from the colors of the blocks on the ends of the top row using the rules for f.

The problem was also solved by:

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