PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2014 Series)

Problem:

A straight wire is bent in a right angle and fixed in place. Two identical beads which slide along the wire are at time zero at rest on the two legs of the wire at perhaps different distances from the vertex. The beads attract each other with equal positive force, directed on the line between them, which may vary smoothly with time. There are no other forces. Prove that the beads arrive at the vertex at the same time.

Solution 1 : (by Bennett Marsh, Senior, Physics & Math, Purdue University)

Let $x(t)$ be the distance of the first bead from the vertex and $y(t)$ be that of the second. Define $\theta(t) = \tan^{-1}(y/x)$ to be the angle between the line of force and one of the legs of the wire. The beads' motion can be described by the differential equations

$$\ddot{x} = -f(t)\cos\theta$$
$$\ddot{y} = -f(t)\sin\theta.$$

It can be checked by substitution that the following solves these equations with the correct initial conditions, where $\theta(t) = \theta(0) =$ constant and r_0 is the initial distance between the beads.

$$x(t) = r_0\cos\theta - \cos\theta\int_0^t\int_0^{t'} f(t'')dt''dt'$$
$$y(t) = r_0\sin\theta - \sin\theta\int_0^t\int_0^{t'} f(t'')dt''dt'.$$

We see then that we always have $y(t) = x(t)\tan\theta$, so that $y = x = 0$ at the same time T.

Solution 2 : (by Steven Landy, Physics Faculty, IUPUI)

Suppose one particle is on the x axis, the other on the y axis. As regards the two particle system, the only external forces are due to the constraints. These are $-F \cdot y/r$ on the x-axis particle and $-F \cdot x/r$ on the y-axis particle. The net force on the system always points from the center of mass toward the origin. Therefore, since the cm velocity is initially zero, the center of mass moves on a straight line to the origin making the two particles arrive simultaneously.

Solution 3 : (by Sachin Kalia, Graduate Student, U of Minnesota))

Let the two sides of the wire be along the x and y axis respectively. Also let x and y denote the instantaneous position of beads (each assumed to have mass m) along the x and y axis respectively. Since the beads are constrained to move along the wire, the coupled motion of equation for them is

$$m\frac{d^2x}{dt^2} = \frac{F(t)x}{\sqrt{x^2+y^2}} \qquad (1)$$

$$m\frac{d^2y}{dt^2} = \frac{F(t)y}{\sqrt{x^2+y^2}} \qquad (2)$$

where $F(t)$ is the smooth time varying force of attraction between the beads. Multiplying (1) and (2) by y and x respectively we get $y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2} = \frac{d}{dt}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right) = 0$. Therefore we must have $yv_x - xv_y =$ constant, where v_x and v_y are the instantaneous velocities of the beads along the x and y axis respectively. Since the relationship holds at all times and at $t = 0$ the beads were at rest, the constant must equal 0. Therefore we can rewrite the previous equation as $\frac{d}{dt}\left(\frac{x}{y}\right) \times y^2 = 0$ or $\frac{x}{y}$ remains constant; therefore the two beads must reach the vertex at the same time. Hence proved.

The problem was also solved by:

Undergraduates: Adam Kline (Fr. Chem/Phys), Yang Mo (So. Phys)

Graduates: Kuang-Ru Wu (Math)

Others: Marco Biagini (Math Teacher, Italy), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Hubert Desprez (Paris, France), Andrew Garmon (Christopher Newport University alumni), Rick Shilling & Bruce Grayson (Orlando, FL), Kipp Johnson (Valley Catholic HS teacher, Oregon), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Matthew Lim, Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Mehtaab Sawhney (HS Student, Commack HS, NY), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Richland Community College, IL), Bjorn Vermeersch (Postdoc, Purdue Univ.)