PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2001 Series)

Problem: Let S_n be the sum of lengths of all the sides and all the diagonals of a regular n-gon inscribed in a unit circle. Evaluate S_n. Find $\lim_{n\to\infty} S_n/n$.

Solution (by Aditya S. Utturwar, Grad. AE, Georgia Inst. Tech., edited by the Panel)

One concludes from geometry that

$$S_n = n\sum_{k=1}^{n-1} \sin k\theta, \quad \theta = \pi/n.$$

Using the identity $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$, we have

$$\begin{aligned}
&2\sin\theta(\sin\theta + \sin 2\theta + \cdots + \sin(n-1)\theta) \\
&= (\cos 0 - \cos 2\theta) + (\cos\theta - \cos 3\theta) + (\cos 2\theta - \cos 4\theta) + \cdots + (\cos(n-2)\theta - \cos n\theta) \\
&= \cos 0 + \cos\theta - \cos(n-1)\theta - \cos n\theta \\
&= 1 + \cos\theta + \cos\theta - (-1) \\
&= 2(1+\cos\theta).
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{S_n}{n} &= \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \cot(\theta/2), \quad \text{so that} \\
S_n &= n\cot(\pi/2n) \quad \text{and} \\
\lim\frac{S_n}{n} &= \lim\cot(\pi/2n) = \infty.
\end{aligned}$$

<u>Remark</u>. The Problem was supposed to ask for $\lim S_n/n^2$. By the above

$$\lim\frac{S_n}{n^2} = \lim\frac{\cos(\pi/2n)}{n\sin(\pi/2n)} = \lim\frac{1}{(\pi/2)(2n/\pi)\sin(\pi/2n)} = \frac{2}{\pi}$$

Complete or partial solutions were received also from:

<u>Undergraduates</u>: Stevie Schraudner (Jr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)