PROBLEM OF THE WEEK
Solution of Problem No. 14 (Spring 2001 Series)

Problem: Suppose f and g are positive-valued piecewise continuous functions defined on the closed interval $[0,1]$ such that $\int_{[0,1]} f = \int_{[0,1]} g = 1$. Show that there exists a subinterval J of $[0,1]$ for which $\int_J f = \int_J g = \frac{1}{2}$.

Solution (by the Panel)

Since f is strictly positive, there exists a unique number A, $0 < A < 1$, such that

$$\int_0^A f = \int_A^1 f = \frac{1}{2}.$$

For every a, $0 \le a \le A$, there exists $b = b(a)$, $A \le b \le 1$ such that $\int_a^b f = \frac{1}{2}$. In particular, $b(0) = A$ and $b(A) = 1$.

Put

$$G(a) = \int_a^{b(a)} g$$

G is continuous and $0 < G(a) < 1$. If $G(0) = \frac{1}{2}$, then $\int_0^A f = \int_0^A g = \frac{1}{2}$, the sought interval is $J = [0, A]$. Assume $G(0) = \int_0^A g < \frac{1}{2}$; then $G(A) = \int_A^1 g > \frac{1}{2}$. By the Intermediate Value Theorem, there is some a_x (unique) such that

$$G(a_x) = \int_{a_x}^{b(a_x)} g = \frac{1}{2}.$$

The sought interval is then $J = [a_{(x)} b_{(a_x)}]$. The case $G(0) > \frac{1}{2}$ is resolved in the same way.

Solved by

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One incorrect solution was received.