PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2002 Series)

Problem: a) Let S_1 be a set of four points in a plane, no three collinear. Show that the four triangles with vertices in S_1 may have equal areas.

b) Let S_2 be a set of five points in space, no four coplanar. Show that the five tetrahedra with vertices in S_2 cannot have equal volumes.

Solution (by the Panel)

a) In a plane arrange the points as vertices of a square.

b) Make an affine transformation of the five points putting them in the x, y, z coordinate space in the positions

$$O : (0, 0, 0), \quad A : (1, 0, 0), \quad B : (0, 1, 0), \quad C : (0, 0, 1), \quad P : (x, y, z).$$

The transformation preserves ratios of volumes, so equal volumes remain equal. Assume

$$|OABP| = |OBCP| = |OCAP|. \tag{1}$$

Since the areas of $\triangle OAB$, $\triangle OBC$, $\triangle OCA$ are equal, P must be on the line $x = y = z$, say $P : (a, a, a)$. If $|OABC| = |PABC|$ the altitudes of these tetrahedra to their common base are equal. Thus $a = 1$ (if $a = 0$, O, P, A, B are coplanar). These tetrahedra are congruent and each has half the volume of the polyhedron $OABCP$.

The three tetrahedra in (1) which have OP as a shared edge make up the interior of $OABCP$ and thus the volume of each is one third the volume of $OABCP$. Therefore the five tetrahedra determined by five points in space cannot all have the same volume.

Solved by:

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

Three incorrect solutions were received.

Several solutions of previous problems were either picked up after noon of the due day or, through our error, were declared late though on time. The correct ones are noted below and will be recorded as on time.

Problem 5: <u>Undergraduates</u>: Stevie Schraudner (Sr, CS/MA), Davis Soetarso (Fr. S), Eric Tkaczyk (Jr, EE/MA)

<u>Graduates</u>: Sravanthi Konduri (CE), K. H. Sarma (NUCL)

<u>Faculty</u>: Fabio Augusto Milner (MA)

<u>Others</u>: Prithwijit De (Un. Cork, Eire)

Problem 7: <u>Graduates</u>: Dan Prater (AAE), K. H. Sarma (NUCL)