Problem: A point $P$ is chosen at random with respect to the uniform distribution in an equilateral triangle $T$. What is the probability that there is a point $Q$ in $T$ whose distance from $P$ is larger than the altitude of $T$? (The answer can be found without integration.)

Solution (by the Panel)

Let $AOB$ be the vertices of $T$, $M$ the midpoint of $OB$, $C$ the orthocenter of $T$, and $R$ the intersection of the altitude of $AB$ (say of length $h$) and the circle with center $A$ and radius $AM$. The sought probability is

$$p = \frac{6 \text{area} (ORM)}{\frac{1}{4} \sqrt{3}}$$

if 1 is the length of the side of $T$.

On taking origin at $O$, positive $x$-axis along $OB$, and positive $y$-axis through $O$ in the direction of $MA$, the coordinates $x, y$ of $Q$ satisfy

$$y = \frac{x}{3}\sqrt{3}, \quad (x - \frac{1}{2})^2 + (y - \frac{1}{2}\sqrt{3})^2 = \frac{3}{4}.$$ 

One finds $x = \frac{3}{4} - \frac{1}{4}\sqrt{6}, y = \frac{1}{4}\sqrt{3} - \frac{1}{4}\sqrt{2}$.

Now $|ORM| = |OCM| - |RCM| = |OCM| - (|RAM| - |RAC|)$, where $|OCM| = \frac{1}{24}\sqrt{3}, |RAC| = \frac{1}{2}\frac{1}{3}\sqrt{3}(\frac{1}{2} - x) = \frac{1}{8}\sqrt{2} - \frac{1}{24}\sqrt{3},$

$|RAM| = \frac{h^2}{2} \sin^{-1}\left(\frac{\frac{1}{2} - x}{h}\right) = \frac{3}{8} \sin^{-1}\left(\frac{\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}}{h}\right)$. 

Hence

$$|ORM| = \frac{1}{24}\sqrt{3} - \frac{3}{8}\sin^{-1}\left(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}\right) + \frac{1}{8}\sqrt{2} - \frac{1}{24}\sqrt{3},$$

$$= \frac{1}{8}\sqrt{2} - \frac{3}{8}\sin^{-1}\left(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}\right),$$

and so

$$p = \sqrt{6} - 3\sqrt{3}\sin^{-1}\left(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}\right) = 0.2067.$$ 

Also solved by:

Graduates: Michael Igarta (ECE)

Others: Regis J. Serinko (PhD, State Coll., PA)

One incorrect solution was received.