PROBLEM OF THE WEEK Solution of Problem No. 10 (Spring 2005 Series)

Problem: Given that $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$, show that for any odd positive integer k there is a rational number r_k such that $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^k dx = r_k \pi$.

Solution (by the Panel)

Let Df denote df/dx. Integrate by parts n-1 times to get

(1)
$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^n dx = \frac{1}{(n-1)!} \int_{-\infty}^{\infty} \frac{D^{n-1} \sin^n x}{x} dx.$$

Note that the integral on the left-hand side is absolutely convergent for $n \ge 2$, and the integrand extends continuously at x = 0. The integrand in the right-hand side has the same property, and the convergence of that integral follows from the arguments below (and from (1) as well).

One can easily show, for example by math. induction, that $\sin^n x$ is a linear combination of $\sin(kx)$, k = 1, ..., n with rational coefficients. Recall that n is odd. It remains to prove our statement for

(2)
$$\int_{-\infty}^{\infty} \frac{D^{n-1}\sin(kx)}{x} \, dx.$$

Since n-1 is even, it is enough to study

(3)
$$\int_{-\infty}^{\infty} \frac{\sin(kx)}{x} dx = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

Now, (1), (2) and (3) prove our statement.

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Update on Problem No. 9:

It was also solved by Didier Alique (France).