PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2005 Series)

Problem: Let $p(x)$ be a continuous function on the interval $[a, b]$, where $a < b$. Let $\lambda > 0$ be fixed. Show that the only solution of the boundary value problem

$$\begin{aligned} &y'' + p(x)y' - \lambda y = 0, \\ &y(a) = y(b) = 0, \quad \text{is} \quad y = 0. \end{aligned}$$

Solution (by the Panel)

Solution I:

The equation easily implies that if y'' exists, then y'' must be continuous, so $y \in C^2([a, b])$. Assume that $y(x)$ is not identically zero on $[a, b]$. Then there is a point in $[a, b]$, where y is either strictly positive or strictly negative. Without loss of generality, we can assume that we have the first case. Then the maximal value of y over $[a, b]$ is positive, and it is attained at a point that is interior for $[a, b]$, let us call it x_0. Then

$$y'(x_0) = 0, \qquad y''(x_0) \leq 0.$$

By the equation, $0 \geq y''(x_0) = \lambda y(x_0) > 0$. This contradiction proves our statement.

Solution II:

Let $q(x) = e^{\int p(x)dx}$ be the "integrating factor" of the equation. Then

$$0 < q, \qquad q' = pq.$$

Multiply the equation by q to get

$$(qy')' - \lambda qy = 0.$$

Multiply by y (or by $\bar{y}$, if complex–valued y's are allowed) and integrate using the boundary conditions. We get

$$-\int_a^b q(y')^2 dx - \lambda \int_a^b qy^2 dx = 0.$$

Since $\lambda > 0$, $q(x) > 0$, this clearly implies $y = 0$.

Solved by:

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Update on Problem No. 10:

Uigvel Pahron (Gran Canarie) was wrongly listed among the people solved Problem No. 10. The participant that actually solved the problem is A. Plaza (ULPGC, Spain).