PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2005 Series)

Problem: Let $x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{\frac{2x_n}{x_n+1}}$.
Prove that

$$\prod_{n=1}^{\infty} x_n = \frac{\pi}{2}.$$

Solution (by Georges Ghosn, Quebec; edited by the Panel)

All terms of the sequence x_n are positive, and therefore we can define the sequence $y_n = \dfrac{1}{x_n}$ by:

$$y_1 = \frac{\sqrt{2}}{2} \quad \text{and} \quad y_{n+1} = \sqrt{\frac{y_{n+1}}{2}}.$$

Using the identity $\cos 2x = 2\cos^2 x - 1$, we get

$$y_1 = \cos\frac{\pi}{2^2}, \quad y_2 = \sqrt{\frac{\cos\frac{\pi}{2^2}+1}{2}} = \sqrt{\frac{2\cos^2\frac{\pi}{2^3}}{2}} = \cos\frac{\pi}{2^3}.$$

By induction, $y_n = \sqrt{\dfrac{\cos\frac{\pi}{2^n}+1}{2}} = \sqrt{\cos^2\dfrac{\pi}{2^{n+1}}} = \cos\dfrac{\pi}{2^{n+1}}$.
Since $\sin x \cos x = \dfrac{1}{2}\sin 2x$, we get

$$\begin{aligned} y_1 y_2 \dots y_n \cdot \sin\frac{\pi}{2^{n+1}} &= y_1 \dots y_{n-1} \cdot \frac{1}{2}\sin\frac{\pi}{2^n} \\ &= \dots = \frac{1}{2^n}\sin\frac{\pi}{2} = \frac{1}{2^n}. \end{aligned}$$

Hence $\qquad x_1 \dots x_n = \dfrac{1}{y_1 \dots y_n} = 2^n \sin\dfrac{\pi}{2^{n+1}} = \dfrac{\pi}{2}\,\dfrac{\sin\left(\frac{\pi}{2^{n+1}}\right)}{\left(\frac{\pi}{2^{n+1}}\right)}.$

Finally $\displaystyle\prod_{n=1}^{\infty} x_n = \lim_{n\to\infty}(x_1 \dots x_n) = \frac{\pi}{2}\cdot\lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{2^{n+1}}\right)}{\frac{\pi}{2^{n+1}}} = \frac{\pi}{2}$
because $\displaystyle\lim_{n\to\infty}\frac{\sin\frac{\pi}{2^{n+1}}}{\frac{\pi}{2^{n+1}}} = \lim_{x\to 0}\frac{\sin x}{x} = 1.$

Also solved by:

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