PROBLEM OF THE WEEK

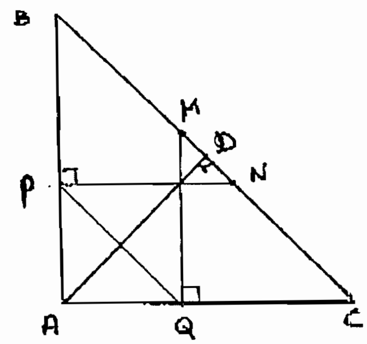
Solution of Problem No. 1 (Spring 2006 Series)

Problem: Let T be the right isosceles triangle of sides $1, 1, \sqrt{2}$. Prove that:

(a) If T is the union of four disjoint sets, then at least one of these sets has diameter $\geq 2 - \sqrt{2}$.

(b) There are four disjoint sets, each of diameter $2 - \sqrt{2}$, whose union is T.

(The diameter of a set is the least upper bound of distances between two points of the set.)



Solution (by Georges Ghosn, Quebec; edited by the Panel)

Consider a right isosceles triangle $T = ABC$ of sides $AB = AC = 1$ and $BC = \sqrt{2}$, D is the midpoint of $[BC]$, and M, N, P, Q are points on $[BC], [BC], [AB]$ and $[AC]$ respectively such that $BM = BP = CN = CQ = 2 - \sqrt{2}$.

(a) Suppose that T is the union of four disjoint sets of diameters less than $2 - \sqrt{2}$. Then A, B, C and D belong to different sets, since $AB = AC = 1$, $BC = \sqrt{2}$, $AD = BD = CD = \dfrac{\sqrt{2}}{2}$ are greater than $2 - \sqrt{2}$.

- M and N belong to the same set as point D since

$$MB = NC = 2 - \sqrt{2}, \quad MC = NB = 2\sqrt{2} - 2 \quad \text{and} \quad MA = NA > AD = \frac{\sqrt{2}}{2}$$

are all greater or equal to $2 - \sqrt{2}$.

- Finally P and Q must belong to the same set as point A since $QC = QM = BP = PN = 2 - \sqrt{2}$ (BPN and QMC are right isosceles triangles) and $QB = PC > AB = 1 > 2 - \sqrt{2}$. But $PQ = AP\sqrt{2} = AQ\sqrt{2} = 2 - \sqrt{2}$ in contradiction with the hypothesis.

(b) The following regions: Triangles APQ, BPM without point P; CQN without point Q; trapezoid $MPQN$ without its sides MP, PQ, QN are four disjoint sets, each of diameter $2 - \sqrt{2}$ and whose union is T because for a triangle the diameter is its longest side, and for a trapezoid the diameter is the longest among its sides and diagonals.

At least partially solved by:

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