PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2007 Series)

Problem: Let $\{a_1, \ldots, a_n\}$ be a permutation of $\{1, \ldots, n\}$, where $n \geq 2$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

Solution (by Georges Ghosn, Quebec; edited by the Panel)

We shall use the famous Cauchy–Schwartz theorem:

$$\left|\sum_{i=1}^{n} c_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} c_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

or

$$\left|\sum_{i=1}^{n} c_i b_i\right|^2 \leq \left(\sum_{i=1}^{n} c_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Let $c_i = \dfrac{\sqrt{a}_i}{i}$, $b_i = \dfrac{1}{\sqrt{a}_i}$. Then we have

$$\left(\sum_{i=1}^{n} \frac{1}{i}\right)^2 \leq \left(\sum_{i=1}^{n} \frac{a_i}{i^2}\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{a_i}\right).$$

Since $\{a_1, \ldots, a_n\}$ is a permetation of $\{1, \ldots, n\}$ we have

$$\sum_{i=1}^{n} \frac{1}{a_i} = \sum_{i=1}^{n} \frac{1}{i} > 0.$$

Cancel this item from both sides of the above inequality, we have

$$\sum_{i=1}^{n} \frac{1}{i} \leq \sum \frac{a_i}{i^2}.$$

Also solved by:

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