PROBLEM OF THE WEEK
Solution of Problem No. 11 (Spring 2008 Series)

Problem: Suppose f and g are non–constant real–valued differentiable functions on $(-\infty, \infty)$. Furthermore suppose

$$f(x+y) = f(x)f(y) - g(x)g(y), \text{ and}$$
$$g(x+y) = f(x)g(y) + g(x)f(y), \text{ for all } x, y.$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

Solution (by Daniel Jiang, Freshman Engineering)

Differentiating both sides of the first equation with respect to x, we get

$$f'(x+y) = f'(x)f(y) - g'(x)g(y)$$

and then letting $x = 0$, letting $g'(0) = k$, and writing f as a function of another variable t,

$$f'(y) = -g'(0)g(y) \Rightarrow f'(t) = -kg(t).$$

Doing the same with the second equation, it is easy to arrive at

$$g'(t) = kf(t).$$

Let $(f(x))^2 + (g(x))^2 = h(x)$. Thus, differentianting with respect to x and substituting:

$$\begin{aligned} 2f(x)f'(x) + 2g(x)g'(x) &= h'(x) \\ 2f(x) \cdot (-kg(x)) + 2g(x) \cdot (kf(x)) &= h'(x) \\ 0 &= h'(x) \end{aligned}$$

The derivative is 0, so h is a constant function (let the constant be C). Using the two given equations, we can compute $h(x+y)$ to be:

$$\begin{aligned} h(x+y) &= f(x+y))^2 + (g(x+y))^2 \\ &= (f(x)f(y) - g(x)g(y))^2 + (f(x)g(y) + g(x)f(y))^2 \\ &= ((f(x))^2 + (g(x))^2)((f(y))^2 + (g(y))^2) \\ &= h(x) \cdot h(y) \end{aligned}$$

We therefore arrive at the equation $C = C^2$, for which there are two solutions, $C = 1$ or $C = 0$. If $C = 0$, then $(f(x))^2 + (g(x))^2 = 0$, but since $(f(x))^2 \geq 0$ and $(g(x))^2 \geq 0$, the only way for this to be true is if $f(x) = g(x) = 0$, which violates the condition that f and g are both non–constant. If $C = 1$, however, no such problems occur, and we have shown that $h(x) = (f(x))^2 + (g(x))^2 = 1$.

Also solved by:

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<u>Update on POW 10</u>: The solution to problem 10 was accidentally published a week early. The list of solutionists has been enlarged.