PROBLEM OF THE WEEK

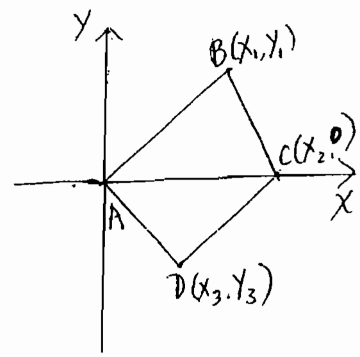
Solution of Problem No. 13 (Spring 2008 Series)

Problem: Prove that a convex quadrilateral is a parallelogram if and only if the centroid of the vertices and the centroid of the area coincide. Here the centroid of the ventices is the center of mass if equal point masses are placed at each vertex, and the centroid of the area is the center of mass if the mass is distributed throughout the interior with constant density.

Solution (by Jinzhong Li, Shaanxi Normal University, China)

Look at the graph on the right. The centroid of the vertices of quadrilateral $ABCD$ is

$$x_{\text{com}} = \frac{1}{4}(x_1 + x_2 + x_3),\ y_{\text{com}} = \frac{1}{4}(y_1 + y_3).$$



The centroid of the area of quadrilateral $ABCD$ is

$$\bar{x}_{\text{com}} = \frac{\iint_{\Delta} x\,dxdy}{\iint_{\Delta} dxdy}, \quad \bar{y}_{\text{com}} = \frac{\iint_{\Delta} y\,dxdy}{\iint_{\Delta} dxdy},$$

where Δ is the region of quadrilateral $ABCD$.

By simple calculation, we find $\bar{x}_{\text{com}} = \dfrac{x_2y_1 + x_1y_1 - x_2y_3 - x_3y_3}{3(y_1 - y_3)}$, $\bar{y}_{\text{com}} = \dfrac{1}{3}(y_1+y_3)$. Now let $x_{\text{com}} = \bar{x}_{\text{com}}$, $y_{\text{com}} = \bar{y}_{\text{com}}$. We have $y_1+y_3 = 0$, $x_1+x_3 = x_2$; i.e., quadrilateral $ABCD$ is a parallelgram.

The converse is obvious. This completes the proof.

Also solved by:

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