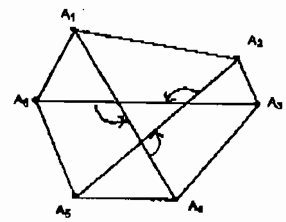
PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2008 Series)

Problem: Suppose a convex hexagon has vertices $A_1, A_2, \ldots, A_6$ in clockwise order and that no side is larger than 1. Show that at least one of the major diagonals is no larger than 2. Here the major diagonals are A_1A_4, A_2A_5, and A_3A_6.



Solution (by Sorin Rubinstein, TAU faculty, Israel)

Let us first remark that

$$\angle(\vec{A_1A_4}, \vec{A_5A_2}) + \angle(\vec{A_5A_2}, \vec{A_3A_6}) + \angle(\vec{A_3A_6}, \vec{A_1A_4}) = 360^\circ \qquad (1)$$

(Here and elsewhere the angles are between 0° and 180°)

This is evident if the major diagonals pass through the same point. If they do not pass through the same point then (1) gives the sum of the exterior angles of the triangle formed by these diagonals. Then, at least one of the summands in (1) must be less then or equal to 120°. We will assume without loss of generality that $\angle(\vec{A_1A_4}, \vec{A_5A_2}) \leq 120^\circ$. (All other cases are obtained by a renumbering of the vertices). We will also assume without loss of generality that $||\vec{A_1A_4}|| \leq ||\vec{A_5A_2}||$ (The other case is treated identically).

From the relation:

$$||\vec{A_1A_4} + \vec{A_5A_2}||^2 = ||\vec{A_1A_4}||^2 + ||\vec{A_5A_2}||^2 + 2||\vec{A_1A_4}|| \cdot ||\vec{A_5A_2}|| \cdot \cos\angle(\vec{A_1A_4}, \vec{A_5A_2})$$

it follows, since $\cos\angle(\vec{A_1A_4}, \vec{A_5A_2}) \geq -\frac{1}{2}$, that:

$$||\vec{A_1A_4} + \vec{A_5A_2}||^2 \geq ||\vec{A_1A_4}||^2 + ||\vec{A_5A_2}||^2 - ||\vec{A_1A_4}|| \cdot ||\vec{A_5A_2}||.$$

From this, since $||\vec{A_1A_4}|| \leq ||\vec{A_5A_2}||$, it follows that

$$||\vec{A_1A_4} + \vec{A_5A_2}||^2 \geq ||\vec{A_1A_4}||^2 + ||\vec{A_5A_2}||^2 - ||\vec{A_5A_2}|| \cdot ||\vec{A_5A_2}|| = ||\vec{A_1A_4}||^2.$$

Thus: $||\vec{A_1A_4}|| \leq ||\vec{A_1A_4} + \vec{A_5A_2}||$ which together with

$$\vec{A_1A_4} + \vec{A_5A_2} = (\vec{A_1A_2} + \vec{A_2A_4}) + (\vec{A_5A_4} + \vec{A_4A_2}) = \vec{A_1A_2} + \vec{A_5A_4}$$

lead to: $||\vec{A_1A_4}|| \leq ||\vec{A_1A_2} + \vec{A_5A_4}|| \leq ||\vec{A_1A_2}|| + ||\vec{A_5A_4}|| \leq 2.$