PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2008 Series)

Problem: Let p be a smooth real–valued function on $\mathbb{R}$ and let y be a solution of the differential equation

$$y''(x) + p(x)y'(x) - y(x) = 0.$$

If y has more than one zero, show that $y(x) \equiv 0$.

Solution (by Sorin Rubinstein, TAU faculty, Israel)

Let $a < b$ be two different zeroes of y. Let x_M and x_m be two numbers in the interval $[a, b]$ for which $y(x_M)$ and $y(x_m)$ are the greatest and the smallest value of y in this interval respectively. Assume that y is not identically zero in the interval $[a, b]$. Then at least one of the numbers $y(x_M)$ and $y(m)$ is not 0.

If $y(x_M) \neq 0$ then $y(x_M) > 0$ and $a < x_M < b$. Moreover, since $(x_M, y(x_M))$ is a local maximum, $y'(x_M) = 0$ and $y''(x_M) \leq 0$. But then $y''(x_M) + p(x_M)y'(x_M) - y(x_M) < 0$, which contradicts the definition of y.

If $y(x_m) \neq 0$ then $y(x_m) < 0$ and $a < x_m < b$. Moreover, since $(x_m, y(x_m))$ is a local minimum, $y'(x_m) = 0$ and $y''(x_m) \geq 0$. But then $y''(x_m) + p(x_m)y'(x_m) - y(x_m) > 0$, which contradicts the definition of y.

Thus y must be identically 0 in the interval $[a, b]$.

Let c be a number such that $a < c < b$. Then clearly $y(c) = y'(c) = 0$. Thus the problem follows from the theorem on the uniqueness of the solution of the second order linear differential equation.

Also solved by:

Graduates: George Hassapis (Math)

Others: Brian Bradie (Christopher Newport U. VA), Mark Crawford (Waubonsee Community College instructor), Elie Ghosn (Montreal, Quebec), Minghua Lin (Shanxi Normal Univ. China)

There were six additional people who correctly proved that y vanishes on an interval but did not prove that y vanishes on the whole line.