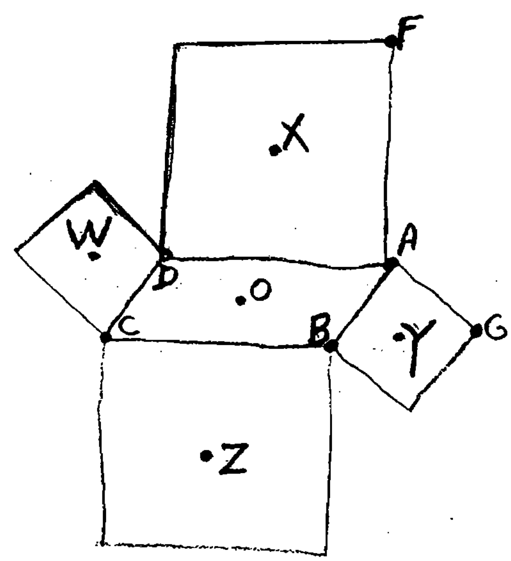
PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2009 Series)

Problem: Show that the centers of the squares erected on the sides of a parallelgram, on the outside the parallelgram, are the vertices of a square.

Solution (by Craig Schroeder, PhD student, Stanford University)

Let the points A, B, C and D be the vertices of a parallelogram centered at the origin. Let R be the linear transformation that rotates vectors counterclockwise by $\pi/2$, and regard the points as vectors. By symmetry, $C = -A$ and $D = -B$. The points F and G are constructed by adding rotated edges to A so $F = A + R(A - D) = A + R(A + B)$ and $G = A + R(B - A)$. The centers of the two squares at A are $X = \frac{1}{2}(D + F) = \frac{1}{2}(A - B) + \frac{1}{2}R(A + B)$ and $Y = \frac{1}{2}(B + G) = \frac{1}{2}(A + B) - \frac{1}{2}R(A - B)$. Observe that $R^2 = -I$ and that $RY = \frac{1}{2}R(A+B) - \frac{1}{2}R^2(A - B) = \frac{1}{2}(A - B) + \frac{1}{2}R(A + B) = X$. Thus, the diagonals of $WXYZ$ are perpendicular and of equal length. Because of the symmetry of the problem, $W = -Y$ and $Z = -X$, so that the diagonals XZ and WY bisect each other at the origin. This makes the quadralateral a parallelogram. Since the diagonals are perpendicular, it is a rhombus. Since they are of equal length, it is a rectangle. Since it is both a rhombus and a rectangle, it is a square.



The problem was also solved by:

Undergraduates: Andy Bohn (Jr. Phys), Michael Burkhart (So. Econ.), David Elden (So. Mech. Engr), Xingyi Qin (Sr. Actuarial Sci.), David White (So. ChE.)

Graduates: Richard Eden (Math), Phuong Thanh Tran (ECE), Tairan Yuwen

Others: Brian Bradie (Christopher Newport U. VA), Melanie Chestnut (Warren Central HS), Gruian Cornel (IT, Romania), Mark Crawford (Waubonsee Community College instructor), Randin Divelbiss (University of Wisconsin–Wausau), Tom Engelsman, Elie Ghosn (Montreal, Quebec), Mike Gloudemans (Grade 10, Biship Dwenger HS, IN), Tigran Hakobyan (Armenia), Chun-Hao Huang (Grad student, National Central Univ. Taiwan), Michael Hudgins (Warren Central HS), S. Kirshanthan (St. Anthony's College, Sri Lanka), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Kamran Najibfard (San Antonio College), Peter Pang (Sophomore, Univ. of Toronto), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Peyman Tavallali (Grad. student, NTU, Singapore), Sheng Xu (SMU)