PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2010 Series)

Problem: Let A be any $n \times n$ matrix in which every entry is either $+1$ or -1.

Prove that the determinant of A is divisible by 2^{n-1}.

Solution (by Tairan Yuwen, Graduate student, Chemistry, Purdue University)

In order to calculate the determinant of A let's add its first row to the other $(n-1)$ rows, and we'll get a new matrix B. According to the property of matrix determinant, we'll have $\det(B) = \det(A)$. Since the elements of A are either $+1$ or -1, the lower $(n-1) \times n$ submatrix of B may only contain $+2, 0$, or -2.

Now let's try to calculate the determinant of B, and we can do that by expanding its first row:

$$\det(B) = \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \det(C_{1j}), \qquad (1)$$

where C_{1j} is the $(n-1) \times (n-1)$ submatrix excluding the first row and the j'th column.

Since all elements of C_{1j} are $+2, 0$ or -2, it means $\det(C_{1j})$ can only have one of the following values: $2^{n-1}, 0$ or -2^{n-1}, which are all divisible by 2^{n-1}. Then according to Equation (1), we'll have $2^{n-1} | \det(B)$.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Eric Haengel (So. Math & Physics), Jason LaJeunesse (So. Mech. Engr), Abram Magner (Jr, CS & Math), Yixin Wang (Fr.)

Graduates: Rodrigo Ferraz de Andrade (Math)

Others: Neacsu Adrian (Romania), Muhammad Ahsan (Pakistan), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Jim Carrubba (Science teacher, IL), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) William Wolber Jr. (ITaP) William Wu (Grad. student, Stanford Univ.)