PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2012 Series)

Problem: Let f be a continuous function on the unit circle which has average value one, that is if θ is as in polar coordinates then $\int_0^{2\pi} \mathbf{f}(\cos(\theta), \sin(\theta))\mathbf{d}\theta = \mathbf{2}\pi$. Show there is an arc of the unit circle of length less than $\mathbf{2}\pi$ such that the average value of f on the arc is one.

Solution 1: (by Craig Schroeder, Post doc, UCLA)

Let

$$g(\phi) = \int_\phi^{\phi+\pi} f(\cos(\theta), \sin(\theta))\, d\theta.$$

Then, $g(\phi) + g(\phi + \pi) = 2\pi$ for all ϕ. It suffices to show that $g(\phi) = \pi$ for some ϕ. Assume WLOG that $g(0) > \pi$, so that $g(\pi) < \pi$. Finally, the results follows from continuity of g and the mean value theorem.

Solution 2: (by Seongjun Choi, Sr. Math, Purdue University)

Let $F(x) = \int_0^x f(\cos\theta, \sin\theta)d\theta$. If $F(x) = x$ for all x in $[0, 2\pi]$ then any arc will do.

If not, any straight line of slope 1 which is not the line $y = x$ and which contains a point (r, k) such that either $r < k < F(r)$ or $F(r) < k < r$ will intersect the graph of F at at least two points $(r_1, F(r_1))$ and $(r_2, F(r_2))$. For $0 < r_1 < r_2 < 2\pi$, since $F(0) = 0$ and $F(2\pi) = 2\pi$. In this case $F(r_2) - F(r_1) = r_2 - r_1$, and the arc $r_1 \le \theta \le r_2$ will do.

The problem was also solved by:

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