PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2012 Series)

Problem: Suppose $f(x)$ is an infinitely differentiable function on $(0,1)$ and continuous on $[0,1]$ and satisfies $f(0) = f(1) = 0$. Prove there is an x in $(0,1)$ such that $f(x) = f'(x)$.

Solution 1: (by Seongjun Choi, Senior, Math, Purdue University)

1) $f(x)$ has an absolute positive maximum on $(0,1)$ or an absolute negative minimum on $(0,1)$ or $f(x) = 0$ for all x. The last case is trivial and the other two can be treated similarly.
2) Therefore we may assume $f(x)$ has an absolute positive maximum on $(0,1)$. Say $f(x)$ achieves its maximum at $c_1 \in (0,1)$. Then, $f'(c_1) = 0$ and $f(c_1) > 0 \implies f(c_1) - f'(c_1) \geq 0$. By the mean value theorem, there exists some point $c_2 \in (0, c_1)$ such that
$$\frac{f(c_1) - f(0)}{c_1} = f'(c_2).$$
Since $c_1 < 1$, $f(0) = 0$, we have $f'(c_2) > f(c_1)$. Also $f(c_1) \geq f(c_2)$. This means
$$f(c_1) - f'(c_1) \geq 0 \qquad f(c_2) - f'(c_2) < 0$$
$f(x) - f'(x)$ is continuous, thus $\exists\, x \in [c_2, c_1]$ such that $f(x) - f'(x) = 0$, as desired.

Solution 2: (by Mingyu Li, Junior, Purdue University)

Set $g(x) = f(x)e^{-x}$. Because $g(x)$ is an infinitely differentiable on $(0,1)$ and continuous on $[0,1]$, we can use the mean value theorem

$$\begin{aligned} \exists x_0 \in (0,1) \quad \left(f(x)e^{-x}\right)\Big|_{x=x_0} &= \frac{f(1)e^{-1} - f(0)e^0}{1-0} = \frac{0-0}{1} = 0 \\ &\Rightarrow f'(x_0)e^{-x_0} - e^{-x_0}f(x_0) = 0 \\ &\Rightarrow f'(x_0) = f(x_0) \end{aligned}$$

so $\exists x_0 \in (0,1) \quad f'(x_0) = f(x_0)$.

The problem was also solved by:

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