PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2012 Series)

Problem: Let $p(x, y)$ be a polynomial in x and y with real coefficients. Suppose $p(x, y) = 0$ for every (x, y) satisfying $x^2 + y^2 = 1$. Show $p(x, y)$ is divisible by $x^2 + y^2 - 1$.

Solution: (by Sorin Rubinstein, Tel Aviv, Israel)

We regard $p(x, y)$ as a polynomial in the variable x over the ring $R[y]$. (i.e. $p(x, y) \in R[y][x]$). Since $x^2 + y^2 - 1$ is a monic polynomial over $R[y]$ we can divide $p(x, y)$ by $x^2 + y^2 - 1$ to obtain: $p(x, y) = (x^2 + y^2 - 1)q(x, y) + m(y)x + n(y)$ for some polynomials $q(x, y), m(y), n(y)$. It follows that $m(y)x + n(y)$ equals zero for every (x, y) satisfying $x^2 + y^2 - 1 = 0$. Then, for every $y \in (0, 1)$ the following equalities hold true:

$$m(y)\sqrt{1 - y^2} + n(y) = 0$$

$$m(y) \cdot \Big(- \sqrt{1 - y^2}\Big) + n(y) = 0$$

We add and subtract these equalities and obtain that $n(y) = 0$ and $m(y) = 0$. Since this is true for every $y \in (0, 1)$, it turns out that the polynomials $m(y)$ and $n(y)$ are identically zero.

Hence $p(x, y) = (x^2 + y^2 - 1)q(x, y)$ meaning that $p(x, y)$ is divisible by $x^2 + y^2 - 1$.

The problem was also solved by:

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