PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2012 Series)

Problem: Prove that for all positive integers n the equations $x^2+y^2=2n$ and $x^2+y^2=n$ have the same number of integer solutions.

Solution: (by Sorin Rubinstein, Rama 22, Tel Aviv, Israel)

We define the function $f : R^2 \to R^2$ by $f(x, y) = (x + y, x - y)$. Then f is invertible and its inverse is $f^{-1}(x, y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$. Let n be a positive integer. If (x_0, y_0) is an integer solution of $x^2 + y^2 = n$, then $f(x_0, y_0)$ is an integer solution of $x^2 + y^2 = 2n$. Indeed: $(x_0 + y_0)^2 + (x_0 - y_0)^2 = 2({x_0}^2 + {y_0}^2) = 2n$. Conversely, if (x_0, y_0) is an integer solution of $x^2 + y^2 = 2n$ then $x_0 = y_0 \mod 2$ – otherwise ${x_0}^2 + {y_0}^2$ would be odd - and $f^{-1}(x_0, y_0)$ is an integer solution of the equation $x^2 + y^2 = n$. Indeed

$$\left(\frac{x_0+y_0}{2}\right)^2 + \left(\frac{x_0-y_0}{2}\right)^2 = \frac{2({x_0}^2+{y_0}^2)}{4} = \frac{2\cdot 2n}{4} = n.$$

It follows that the restriction of $f(x, y)$ to the set of all integer solutions of the equation $x^2 + y^2 = n$ is an one to one correspondence between this set and the set of all integer solutions of the equation $x^2 + y^2 = 2n$. Hence the equations $x^2 + y^2 = n$ and $x^2 + y^2 = 2n$ have the same number of solutions. This number is necessarily finite because any integer solution (x_0, y_0) of the equation $x^2 + y^2 = n$ must satisfy $|x_0| \le \sqrt{n}$ and $|y_0| \le \sqrt{n}$

The problem was also solved by:

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