PROBLEM OF THE WEEK
Solution of Problem No. 6 (Spring 2013 Series)

Problem:

Suppose $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1$ **and that for every real number** r**,**

$$\sum_{\{i:\, a_i > r\}} b_i = 0.$$

Find $\sum_{i=1}^{n} (a_i + b_i)^2$**.**

Solution: (by Chenkai Wang, Sophomore, Mathematics, Purdue University)

Let r be less than the largest a_i but greater than the second largest a_i. (If there is no second largest a_i, that means all a_i are equal, then any r less than the largest one is ok.) Since $\sum_{\{i:a_i>r\}} b_i = 0$, we have $\sum_{\{i:a_i>r\}} a_i b_i = (\max a_i) \sum_{\{i:a_i>r\}} b_i = 0$. For the rest of a_i, continue this process. After we exhaust all a_i, we get $\sum_{i=1}^{n} a_i b_i = 0$. Finally, simple calculation gives us

$$\sum_{i=1}^{n} (a_i + b_i)^2 = \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2 \sum_{i=1}^{n} a_i b_i = 1 + 1 = 2.$$

The problem was also solved by:

Undergraduates: Kilian Cooley (Sr. Math & AAE), Bennett Marsh (So. Engr.)

Graduates: Tairan Yuwen (Chemistry)

Others: Nadir Amaioua (Graduate Student, Ecole Polytechnique, Canada), Siavash Ameli (Graduate Student, IIT, Chicago), Marco Biagini (Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Pierre Castelli (Antibes, France), Jingmin Chen (Graduate Student, Drexel Univ.), Jon Cohen (Graduate Student, UMCP), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Ayush Gupta (Student, IIT, Delhi, India), Sachin Khapli (Professor, N.Y.

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