PROBLEM OF THE WEEK
Solution of Problem No. 1 (Spring 2015 Series)

Problem:

Let $f(x)$ be a strictly increasing continuous function on a bounded interval $[a,b]$. Choose c in $[a,b]$. Consider the two curvilinear triangles bounded by the vertical lines $x=a, x=b$, the horizontal line $y=f(c)$ and the graph of f. For which position c is the sum of the areas of these curvilinear triangles minimal?

Solution by Hubert Desprez, Paris, France

First by $x \longrightarrow \dfrac{x-a}{b-a}$ assume wlog $(a,b)=(0,1)$. We have to minimize
$\varphi(c) = \int_0^c (f(c)-f) + \int_c^1 (f-f(c)) = (2c-1)f(c) + \int_c^1 f + \int_c^0 f$, by monotony,
$\varphi(c) - \varphi(1/2) = 2\left((c-1/2)f(c) - \int_{1/2}^c f\right) \geq 0$, answer is $c=(a+b)/2$.

Remark from the panel: Solutions needed to apply to all strictly increasing functions, not just differentiable ones. For those who did assume differentiability we note that a strictly increasing differentiable function need not have a positive derivative at all points, as x^3 shows.

The problem was also solved by:

Undergraduates: Eric Buedel (Fr. Math), Sameer Manchanda (Fr. CS), Alan Min (Fr.)

Graduates: Cheng Li (STAT), Kuang-Ru Wu (Math)

Others: Manuel Barbero (New York), Chao-yi Chang (Graduate Student, Taiwan), Hong-wei Chen (Professor, Christopher Newport Univ. Virginia), Jiehua Chen (Quantitative Engineering Design Inc.), Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Matthew Lim, Paolo Perfetti (Roma, Italy), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Mehtaab Sawhney (HS Student, Commack HS, NY), David Stoner (HS Student, Aiken, S. Carolina), Jiazhen Tan (HS Student, China) Feng Tian (Studnet, Duke U)