PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2000 Series)

Problem: Let $\triangle$ be an isosceles triangle for which the length of a side and the length of the base are rational. Prove that the radius of the incircle of $\triangle$ is rational if and only if the two right triangles formed by the altitude to the base are similar to a right triangle with integer side lengths.

Solution (by the Panel)

Let r be the length of the radius of the inscribed circle, a the length of the side, $2b$ the length of the base, h the length of the altitude to the base. The area A of the triangle can be expressed in two ways:

$$2A = r(2a + 2b) = 2bh,$$

hence $r = bh/(a + b)$, and r is rational if and only if h is rational.

Now if h is rational, then $\exists n \in \mathbb{N}$ such that an, bn, hn are all in $\mathbb{N}$, and form the sides of a triangle similar to the original triangle with sides a, b, h.

Conversely, if the triangle with sides b, a, h is similar to one with integral sides then $\exists t \in \mathbb{R}$ such that $b^2t^2 = a^2t^2 + h^2t^2; bt, at, ht \in \mathbb{N}$, but a is rational, so t is rational and, therefore, h is rational.

Also solved by:

<u>Undergraduates</u>: Jeffrey D. Moser (Fr. MA/CS), Yee-Ching Yeow (Jr. Math)

<u>Graduates</u>: Gajath Gunatillake (MA), Wook Kim (MA)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

<u>Others</u>: Damir D. Dzhafarov (Sr. Harrison H.S., WL), Mike Hamburg (Jr. St. Joseph's H.S., South Bend)

Late: A correct solution of Problem 7 was mailed on time from Japan, but was received late. The solver was Tetsuji Nishikura, a physician of Hyougo Prefecture.