PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2001 Series)

Problem: For $n = 1, 2, \cdots$, set $S_n = \sum_{k=0}^{3n} \binom{3n}{k}$, $T_n = \sum_{k=0}^{n} \binom{3n}{3k}$. Prove that $|S_n - 3T_n| = 2$.

Solution (by Steven Landy, Faculty, Physics at IUPUI)

Let $w = e^{2\pi i/3}$, so $w^2 = \overline{w}$ (conjugate), $w^3 = 1$, $1 + w + w^2 = 0$. Then

$$S_n = \sum_{k=0}^{n} \binom{3n}{k} 1^k = (1+1)^{3n}$$

by the Binomial Theorem. Also

$$U_n = \sum_{k=0}^{n} \binom{3n}{k} w^k = (1+w)^{3n}$$

and

$$V_n = \sum_{k=0}^{n} \binom{3n}{k} w^{2k} = (1+\overline{w})^{3n}.$$

Because $1^k + w^k + w^{2k} = 0$ unless k is a multiple of 3, when it is $= 3$,

$$S_n + U_n + V_n = 3T_n,$$

and so

$$\begin{aligned} 3T_n - S_n = U_n + V_n &= (1+w)^{3n} + (1+\overline{w})^{3n} = (-w^2)^{3n} + (-\overline{w}^2)^{3n} \\ &= (-1)^n [e^{4\pi i n} + e^{-4\pi i n}] = 2(-1)^n, \end{aligned}$$

so $|S_n - 3T_n| = 2$.

Also solved by:

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