

# Tangencies between holomorphic maps and holomorphic laminations

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Let  $F$  be a holomorphic lamination of codimension one in an open set  $V$  in a complex Banach space  $B$ . In this paper, this means that  $V = W \times \mathbf{C}$ , where  $W$  is a neighborhood of the origin in some Banach space, and the leaves  $L_\lambda$  of the lamination are disjoint graphs of holomorphic functions  $\mathbf{w} \mapsto f(\lambda, \mathbf{w})$ ,  $W \rightarrow \mathbf{C}$ . Here  $\lambda$  is a parameter and we assume that the dependence of  $f$  on  $\lambda$  is continuous. A natural choice of this parameter is such that  $\lambda = f(\lambda, 0)$ , in which case the continuity with respect to  $\lambda$  follows from the so-called  $\lambda$ -lemma of Mane-Sullivan-Sad and Lyubich, see, for example [5]. With this choice of the parameter, our definition of a lamination coincides with that of a holomorphic motion of  $\mathbf{C}$ .

Let  $\gamma : U \rightarrow V$  be a holomorphic map,  $U \subset \mathbf{C}^m$ . We say that  $\gamma$  is tangent to the lamination at a point  $\mathbf{z}_0 \in U$  if the image of the derivative  $\gamma'(\mathbf{z}_0)$  is contained in the tangent space  $T_L(\gamma(\mathbf{z}_0))$ , where  $L$  is the leaf passing through  $\gamma(\mathbf{z}_0)$ . A leaf for which this holds is called a tangent leaf to  $\gamma$ .

**Theorem.** *Let  $K$  be a compact subset of  $U$ . Then the set of leaves tangent to  $\gamma$  at the points of  $K$  is finite.*

For the case of holomorphic curves ( $m = 1$ ) this result is contained in [1, Lemma 9.1] where it is credited to Douady. Artur Avila, in a conversation with the authors, proposed to extend this result to arbitrary holomorphic maps. According to Avila, it has several other applications to holomorphic dynamics.

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*Proof.* We assume without loss of generality that

$$f(0, \mathbf{w}) \equiv 0,$$

and that  $L_0$  is tangent to  $\gamma$  at  $\mathbf{z}_0 = 0$ .

We have to show that other tangent leaves cannot accumulate to  $L_0$ . Suppose the contrary, that is suppose that there is a sequence  $\lambda_k \rightarrow 0$  such that  $L_{\lambda_k}$  are tangent to  $\gamma$ , and let  $L_{\lambda_k}$  be the graphs of the functions  $f_k(\mathbf{w}) = f(\lambda_k, \mathbf{w})$ . We may assume that tangency points  $\mathbf{z}_k \rightarrow 0$ .

We make several preliminary reductions.

1. Let

$$\gamma(\mathbf{z}) = (\phi(\mathbf{z}), \psi(\mathbf{z})) \in W \times \mathbf{C}.$$

Consider the new lamination in  $U \times \mathbf{C}$  whose leaves are the graphs of  $f^*(\lambda, \mathbf{z}) = f(\lambda, \phi(\mathbf{z}))$  and the new map  $\gamma^*(\mathbf{z}) = (\mathbf{z}, \psi(\mathbf{z}))$ . Then  $\gamma^*$  is tangent to a leaf  $L^*$  if and only if  $\gamma$  is tangent to  $L$ . This reduces our problem to the case that  $W$  is an open set in  $\mathbf{C}^m$  and the map  $\gamma$  is a graph of a function  $\psi$  of the same variable as the functions  $f_k$ . From now on we assume that  $U = W$  and  $\gamma(\mathbf{w}) = (\mathbf{w}, \psi(\mathbf{w}))$ .

2. Now we reduce the problem to the case that  $\psi$  is a monomial. For this we use the desingularization theorem of Hironaka [4, 2, 3, 6].

Let  $X$  be a complex analytic manifold, and  $\psi$  an analytic function on  $X$ . Then there exists a complex analytic manifold  $M$  and a proper surjective map  $\pi : M \rightarrow X$  such that the restriction of  $\pi$  onto the complement of the  $\pi$ -preimage of the set  $\{\psi = 0, \psi' = 0\}$  is injective and for each point  $\mathbf{z}_0 \in \pi^{-1}(\{\psi = 0\})$  there is a local coordinate system with the origin at  $\mathbf{z}_0$  such that  $\psi \circ \pi$  is a monomial  $z_1^{m_1} \dots z_n^{m_n}$ .

Let  $Y = W \times \mathbf{C}$ , and let  $S \subset Y$  be the set of points  $(\mathbf{w}, t)$  with  $t \neq 0$  such that the graph of  $t = \psi(\mathbf{w})$  is tangent to the lamination. In our proof by contradiction, we assume that the origin belongs to the closure of  $S$ . Let  $N = M \times \mathbf{C}$ , and let  $\rho : N \rightarrow Y$  be the map defined by  $\rho(\mathbf{z}, t) = (\pi(\mathbf{z}), t)$ . Then  $\rho^{-1}(F)$  is the lamination whose leaves are the components of the  $\rho$ -preimages of the leaves of  $F$ , and the set  $T = \rho^{-1}(S)$  has a limit point  $(\mathbf{z}_0, 0)$  with  $\pi(\mathbf{z}_0) = 0$  since  $\pi$  is proper. Also the set  $T$  is exactly the set of those points in  $N$  where the graph  $t = \psi \circ \pi(\mathbf{z})$  is tangent to the foliation  $\rho^{-1}(F)$  since  $\rho$  is injective in a neighborhood of each point of  $T$ . (Any point  $(\mathbf{z}, t)$  where  $\rho$  is not injective satisfies  $\psi(\pi(\mathbf{z})) = 0$  while at every point of  $T$  we have  $\psi(\pi(\mathbf{z})) \neq 0$ .) This reduces our problem to the case that  $\psi$  is a monomial.

3. We may assume now that  $W = \{\mathbf{z} : |\mathbf{z}| < 2\}$ . So we are in the following situation.

$$\psi(z) = z_1^{m_1} \dots z_n^{m_n},$$

and  $\{f_k\}$  is a family of holomorphic functions on  $W$  with disjoint graphs,  $f_k(\mathbf{z}) \neq 0$  for  $\mathbf{z} \in W$ , and  $f_k \rightarrow 0$  as  $k \rightarrow \infty$  uniformly on  $W$ . Moreover, for some sequence  $\mathbf{z}_k \rightarrow 0$  we have

$$f_k(\mathbf{z}_k) = \prod_{j=1}^n z_{j,k}^{m_j}, \quad (1)$$

$$\text{grad } f_k(\mathbf{z}_k) = (m_1 z_{1,k}^{m_1-1} z_{2,k}^{m_2} \dots, m_2 z_{1,k}^{m_1} z_{2,k}^{m_2-1} \dots, \dots) \quad (2)$$

assuming zero values for the components with  $m_j = 0$ . We may assume that  $|f_k| \leq 1$  in  $W$ . Setting  $f_k = \exp g_k$  we obtain that  $\Re g_k \leq 0$ . Now we put

$$h_k(\mathbf{z}) = g_k(\mathbf{z} + \mathbf{z}_k) - g_k(\mathbf{z}_k).$$

Then the  $h_k$  are defined in the unit ball and satisfy

$$\Re h_k(\mathbf{z}) \leq \sum_{j:m_j>0} m_j \log |z_{j,k}|^{-1}.$$

From this we conclude that

$$\left| \frac{\partial h_k}{\partial z_j}(0) \right| \leq 2 \sum_{j:m_j>0} m_j \log |z_{j,k}|^{-1}. \quad (3)$$

This follows from the

**Lemma.** *Let  $h$  be an analytic function in the unit disc,  $h(0) = 0$ , and  $\Re h \leq A$ , where  $A > 0$ . Then  $|h'(0)| \leq 2A$ .*

This is an immediate consequence of the Schwarz Lemma.

On the other hand, (1) and (2) imply that, for  $m_j > 0$ ,

$$\left| \frac{\partial h_k}{\partial z_j}(0) \right| = \frac{m_j}{|z_{j,k}|}. \quad (4)$$

Assume without loss of generality that  $m_1 > 0$  and

$$|z_{1,k}| = \min_{j:m_j>0} |z_{j,k}|.$$

Then the RHS of (3) is at most

$$\text{const} \log |z_{1,k}|^{-1},$$

while the RHS of (4) is

$$\frac{m_1}{|z_{1,k}|}.$$

As  $|z_{1,k}| \rightarrow 0$ , we obtain a contradiction which proves our theorem.

## References

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