Chapter 10

Differentials/Tangent bundle (Continued)

10.1 Derivative of map

Suppose that $F : R \to S$ is a homomorphism of $k$-algebras. Then there exists an $S$-module homomorphism

$$dF : S \otimes_R \Omega_R \to \Omega_S$$

(10.1)

sending $s \otimes dr \mapsto sd(F(r))$.

**Example 10.1.1.** Let $F : k[y_1, \ldots, y_m] \to k[x_1, \ldots, x_n]$ given by sending $y_i \to f_i(x_1, \ldots, x_n)$. Then from the definition

$$dF(1 \otimes dy_i) = df_i = \sum_j \frac{\partial f_i}{\partial x_j} dx_j$$

Thus

$$S^m \cong S \otimes_R \Omega_R \xrightarrow{dF} \Omega_S \cong S^n$$

is given by the Jacobian matrix.

As the above example makes clear, $dF$ can be thought of as the derivative of $F$.

Suppose $f : X \to Y$ is a map of affine varieties with $p \in X$ and $q = f(p)$. Then let $R$ and $S$ be the local rings at $q$ and $p$ respectively, and $F : R \to S$ the associated homomorphism. Then from proposition 9.2.1, we see that $dF$ induces a map between the tangent spaces $T_p X \to T_q Y$. This can be constructed more directly. Since $F$ is local, it takes the maximal ideal $m_R \subset R$ to $m_S \subset S$. Therefore it induces a linear map

$$m_R/m_R^2 \to m_S/m_S^2$$

The dual map is precisely above map $T_p X \to T_q Y$. 

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10.2 Nonaffine varieties

Let $X$ be an affine variety (or scheme) with coordinate ring $R$. This ring is noetherian in the case of varieties, and we will assume this in the scheme case. Given a finitely generated $R$-module $M$, we can find a surjection $f : R^n \to M$ for $n$. The kernel $\ker f$ is again finitely generated, because $R$ is noetherian. Therefore we can find a surjection $R^m \to \ker f$. We can combine this into an exact sequence

$$R^m \to R^n \to M \to 0$$

called a presentation for $M$. Recall that we have a sheaf of rings $\mathcal{O}_X$ such that for any basic open set

$$\mathcal{O}_X(D(f)) = R[1/f]$$

We note that any element of $r$ gives rise to a morphism $\mathcal{O}_X \to \mathcal{O}_X$ sending to $1 \mapsto r$. We want to extend this to $M$. We can use the presentation to construct a morphism

$$\mathcal{O}_X^m \to \mathcal{O}_X^n$$

We define the presheaf

$$\tilde{M}'(U) = \text{coker } \mathcal{O}_X^m(U) \to \mathcal{O}_X^n(U)$$

We turn this into a sheaf by sheafifying

$$\tilde{M} = (\tilde{M}')^+$$

We have the following properties

**Proposition 10.2.1.**

1. $\tilde{M}$ is a sheaf of $\mathcal{O}_X$-modules, i.e. $\tilde{M}(U)$ is an $\mathcal{O}_X(U)$-module and the module compatible with restriction.

2. $\tilde{M}$ as a sheaf of modules does not depend on the presentation.

3. $\tilde{M}(D(f)) \cong M[1/f]$.

An $\mathcal{O}_X$-module which arises this way is called coherent. In the nonaffine, coherence means that the module is locally of the form $\tilde{M}$ for a finitely generated module. We apply the construction to $\Omega_R$ and $T_R$ to get the sheaf of Kähler differentials

$$\Omega_X = \tilde{\Omega}_R$$

and the tangent sheaf

$$T_X = \tilde{T}_R$$

We can extend these constructions to general varieties or schemes.

**Theorem 10.2.2.** Given a variety $X$ (or scheme over Spec $k$), there exists coherent $\mathcal{O}_X$-modules $\Omega_X$ and $T_X$ which are given by the last two formulas when restricted to an affine open set.
Sketch. Choose a cover \{U_i\} by open affines. The sheaves are obtained by gluing the sheaves \(\Omega_{U_i}, \mathcal{T}_{U_i}\) in the appropriate sense. Most books will use a different construction, however.

Example 10.2.3. Let \(X = \mathbb{P}^1_k\). We use the standard open covering by \(U_0, U_1\) by two affine lines. The first has coordinate \(x\) and the second has coordinate \(y = x^{-1}\). Given a regular differential \(f(x)dx\) on \(U_0\), it transforms to \(-f(y^{-1})y^{-2}dy\) on \(U_1\). Thus we can conclude that there are no nonzero regular differentials on \(\mathbb{P}^1\).

We want to construct an analogue of (10.1). First we replace \(F\) by a regular map \(f : X \to Y\). Given a sheaf \(\mathcal{F}\) on \(Y\), we define the pullback by

\[ f^{-1}\mathcal{F}(U) = \mathcal{F}(f(U)) \]

if \(f(U)\) were open. In general, we approximate by opens and set

\[ f^{-1}\mathcal{F}(U) = \lim_{V \supset f(U)} \mathcal{F}(V) \]

If \(\mathcal{F}\) is an \(\mathcal{O}_Y\)-module, then \(f^{-1}\) is an \(\mathcal{O}_{f^{-1}\mathcal{O}_Y}\)-module. We define the inverse image \(f^*\mathcal{F}\) to be the sheafification of the presheaf

\[ U \mapsto \mathcal{O}_X(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} f^{-1}\mathcal{F}(U) \]

These operations can be understood in terms of their stalks

Lemma 10.2.4. For any \(p \in X\), the stalks

\[ f^{-1}\mathcal{F}_p \cong \mathcal{F}_{f(p)} \]

\[ f^*\mathcal{F}_p \cong \mathcal{O}_{X,p} \otimes \mathcal{O}_{Y,f(p)} \mathcal{F}_{f(p)} \]

Now we can define what we were after.

Proposition 10.2.5. Given a regular map of varieties \(f : X \to Y\), there exists a morphism \(f^*\Omega_Y \to \Omega_X\), such that on stalks, it coincides with the map

\[ \mathcal{O}_{X,p} \otimes \Omega_{Y,f(p)} \to \Omega_{X,p} \]

given by (10.1).

10.3 The genus of a curve

Let \(X\) be a nonsingular projective curve over an algebraically closed field \(k\). The genus of \(X\) is the dimension

\[ g = \dim_k \Omega^1_X(X) \]
It is not a priori clear that this number is finite, but it is. From the last section, we know that the genus of $\mathbb{P}^1$ is zero. To facilitate computation of more examples, we give an alternative. Let $K = k(X)$ be the field of rational functions on $X$. This is a field of transcendence degree 1. Given a point $p \in X$, we can form the local ring $O_p = O_{X,p} \subset K$. Since $X$ is a curve, $\dim O_p = 1$. Since $X$ is nonsingular, $\dim m/m^2 = 1$. This implies (by Nakayama’s lemma) that $m$ is principal. Therefore $O_p$ is a discrete valuation ring. This means that there is a function

$$ord_p : K \to \mathbb{Z} \cup \{\infty\}$$

such that

1. $ord_p : K^* \to \mathbb{Z}$ is a surjective homomorphism.
2. $ord_p(0) = \infty$
3. $ord_p(f + g) \geq \min(\text{ord}_p(f), \text{ord}_p(g))$
4. $f \in O_p$ if and only if $\text{ord}_p(f) \geq 0$.

A function satisfying (1), (2), (3) is called a discrete valuation. (Surjectivity is not usually required, but it simplifies the story.) We point out the following useful fact. Although we won’t actually need it.

**Theorem 10.3.1.** Every discrete valuation arises, as above, from a unique $p \in X$.

We can form the $K$-vector space $\Omega_K = \Omega^1_{K/k}$. This is can be seen to be one dimensional. It is spanned by $df$ for any nonconstant $f \in K$. We want to extend $ord_p$ to $\Omega_K$. First choose a generator $x$ for $m \subset O_p$, called a local parameter at $p$. If $\omega = fx \in K^*$, define

$$ord_p \omega = ord_p(f)$$

So in particular, $ord_p dx = 0$.

**Lemma 10.3.2.** This is well defined.

**Proof.** If $x'$ is another local parameter, we can write $x' = ux$, with $ord_p u = 0$. Then $ord_p (dx') = ord_p (udx + xdu) = 0$. \qed

We define the space of regular differentials by $\{\omega \in \Omega_K \mid \forall p \in X, ord_p \omega \geq 0\}$

**Lemma 10.3.3.** $\Omega^1_X(X)$ is precisely the space of regular differentials.

We are now ready to do a basic example. Assume that $\text{char } k \neq 2$. Choose $2g + 2$ distinct points $a_i \in \mathbb{A}^1_k$. Let $C_1 \subset \mathbb{A}^2_k$ be defined by

$$y^2 = \prod (x - a_i)$$
We want to complete this to a nonsingular projective curve $C$. The closure in $\mathbb{P}^2_k$ is usually singular, so instead we use a gluing construction. Let $C_2 \subset \mathbb{A}^2_k$ be given by
\[
Y^2 = \prod (1 - a_iX)
\]
We glue $C = C_2 \cup C_1$ by identifying $X = x^{-1}$ and $Y = yx^{-g-1}$. $C$ is called a hyperelliptic curve, although technically this name is reserved for when $g > 1$. We have a regular map $C \to \mathbb{P}^1$ given by projection to the $x$-axis for $C_1$. This map is 2 to 1 everywhere except over the $a_i$'s. These are called branch, or ramification, points. In algebraic terms, we have a degree two extension $k(\mathbb{P}^1) \subset K = k(C)$.

**Theorem 10.3.4.** When $g > 0$, $\omega = \frac{dx}{y} \in \Omega_K$ is regular.

**Proof.** We just have to check that $\text{ord}_p \omega \geq 0$ for all $p \in C$. It suffices to do this for $p$ lying over $a_i$ or $\infty$. In the first case, we can assume with loss of generality that $i = 1$, and $a_1 = 0$. Let $u = (x - a_2)(x - a_3) \ldots$. Then
\[
\text{ord}_0 \frac{dx}{y} = \text{ord}_0 \left( \frac{2dy}{u} - \frac{xdu}{yu} \right) \geq 0
\]
There are two points over $\infty$ given by to $X = 0$, $Y = \pm 1$. Denote these by $0'$, $0''$. In algebraic terms, we have an inclusion of local rings $\mathcal{O}_\infty \subset \mathcal{O}_{0'}$, and the local parameter $X$ in the first ring extends to a local parameter of the second. The same holds for the second point. In these coordinates
\[
\omega = - \frac{X^{g-1}}{Y} dX
\]
This has nonnegative valuation at these points because $g > 0$.

**Corollary 10.3.5.** If $g > 0$, $C$ has positive genus. In particular, $C$ is not isomorphic to $\mathbb{P}^1_k$.

A more detailed analysis will show that
\[
\Omega^1_C(C) = \{ f(x)\omega \mid f(x) \text{ is a polynomial of } \deg < g \}
\]
Therefore the genus of $C$ is exactly $g$. Some details will be given as exercises.

### 10.4 Exercises

**Exercise 10.4.1.**

1. Determine all homomorphisms $F : k[x] \to k[x]$ such that $dF$ is identically 0. The answer depends on char $k$.

\[^1\text{Technically, branch points are downstairs, and ramification points are upstairs. Although}
\text{some people interchange these terms.}\]
2. Give examples of regular maps from \( f : \mathbb{P}^1_k \to \mathbb{P}^1_k \) such that natural map \( f^*\Omega_{\mathbb{P}^1_k} \to \Omega_{\mathbb{P}^1_k} \) is zero.

3. Given an algebraic group \( G \), with \( g \in G \), define the regular map \( L_g : G \to G \) by \( L_g(h) = gh \). A differential \( \omega \in \Omega_G(G) \) is called left invariant if \( L_g^*\omega = \omega \) for all \( g \in G \). Determine the left invariant differentials for the multiplicative group \( G = G_m = \mathbb{A}^1_k - 0 \).

4. Show that for the hyperelliptic curve \( C \) constructed above, \( f(x) \frac{dx}{y} \) is regular if \( \deg f(x) < g \).