NOTES ON D-MODULES AND CONNECTIONS WITH HODGE THEORY

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These notes are almost entirely expository and result from my attempt to learn this material. The first part summarizes D-module theory up to Riemann-Hilbert. The second part discusses vanishing cycles in its various forms. This is needed in the next part which summarizes the basics of Morihiko Saito's theory of Hodge modules. My main motivation for going through all this was to convince myself that Saito's methods and the more naive construction in [A] yield the same mixed Hodge structure on the cohomology of a geometric variation of Hodge structure. The proof of this is given in part 4. Readers interested in just this part, may find the note [A2] on the"ArXiv" more convenient.

I gave some informal talks on this material at KIAS in Seoul in 2005 and TIFR Mumbai in 2008. I would them for giving me the opportunity to do so.

Contents

1. D-modules

1.1. **Weyl Algebra.** Fix a positive integer n. The nth Weyl algebra D_n over $\mathbb{C}$ is the ring of differential operators with complex polynomial coefficients in n variables. More formally, D_n can be defined as the *noncommutative* $\mathbb{C}$-algebra generated by symbols $x_1, \dots x_n$, $\partial_1 = \frac{\partial}{\partial x_1}, \dots \partial_n = \frac{\partial}{\partial x_n}$ subject to relations

$$x_i x_j = x_j x_i$$
$$\partial_i \partial_j = \partial_j \partial_i$$
$$\partial_i x_j = x_j \partial_i, \text{ if } i \neq j$$
$$\partial_i x_i = x_i \partial_i + 1$$

The last two relations stem from the Leibnitz rule $\partial_i(x_j f) = \partial_i(x_j) f + x_j \partial_i f$. These relations can be expressed more succinctly, using commutators as

$$[x_i, x_j] = [\partial_i, \partial_j] = 0$$
$$[\partial_i, x_j] = \delta_{ij}$$

There is a sense in which D_n is almost commutative that I want to explain. From the defining relations, it follows that any $P \in D_n$ can be expanded uniquely as

$$P = \sum \alpha_{I,J} x^I \partial^J$$

where $I, J \in \mathbb{N}^n$, $x^I = x_1^{I_1} \dots x_n^{I_n}$ etc. The maximum value of $J_1 + \dots J_n$ occurring in this sum is the *order* of P. We write $F_k D_n$ for the space of operators of order at most k. It is easy to see that $F_k F_m \subseteq F_{k+m}$. Thus the associated graded

$$Gr(D_n) = \bigoplus_k F_k / F_{k-1}$$

inherits a graded algebra structure.

Lemma 1.2. *Given $P, Q \in D_n$, we have $order([P,Q]) < order(P) + order(Q)$*

Sketch. It's enough to check this when P, Q are monomials, i.e. expressions of the form $x^I \partial^J$. In this case, it is a straight forward consequence of induction and the defining relations. □

Corollary 1.3. *$Gr(D_n)$ is commutative.*

Slightly more work yields:

Theorem 1.4. *$Gr(D_n)$ is isomorphic to the polynomial ring $\mathbb{C}[x_1, \dots x_n, \xi_1, \dots \xi_n] = R_{2n}$*

I want to sketch a slightly nonstandard proof of this. First "quantize" D_n to obtain a ring H_n which has an additional variable q subject to the relations that q commutes with x_i and ∂_j and

$$[\partial_i, x_j] = q\delta_{ij}$$

The remaining relations are the same as for D_n: the x's and ∂'s commute among themselves. I will call H_n the Heisenberg algebra, since it nothing but the universal enveloping algebra of the Heisenberg Lie algebra. We see from the relations that

$D_n = H_n/(q-1)$ and the "classical limit" $H_n/(q)$ is the polynomial ring R_{2n}, where $(q-\lambda)$ is the two sided ideal generated by this element.

Now form the Rees algebra

$$Rees = \bigoplus t^k F_k \subset \mathbb{C}[t] \otimes D_n$$

with t a central element. The theorem will follow from the next result which is straightforward.

Lemma 1.5.
(1) *$Gr(D_n) \cong Rees/(t)$*
(2) *The map $Rees \to H_n$ sending $x_i \mapsto x_i$, $t\partial_j \mapsto \partial_j$ and $t \mapsto q$ is an isomorphism.*

In more geometric terms, we have an identification between $Gr(D_n)$ and the ring of polynomial functions on the cotangent bundle $T^*\mathbb{C}^n$.

1.6. **D_n-modules.** The notion of a D_n-module gives an abstract way to think about systems of linear partial differential equations in n-variables. Since the ring D_n is noncommutative, we have to be careful about distinguishing between left-modules and right D_n-modules. I will often be lazy, and refer to a left D_n-module simply as a D_n-module. Here are some examples.

Example 1.7. *D_n is automatically both a left and right D_n-module.*

Example 1.8. *Let $R_n = \mathbb{C}[x_1, \dots x_n]$ be the polynomial ring. This is a left D_n-module where x_i acts by multiplication, and ∂_i by $\frac{\partial}{\partial x_i}$.*

Example 1.9. *Given operators $P_1 \dots P_N \in D_n$, the left (resp. right) ideal $\sum D_n P_i$ (resp. $\sum P_i D_n$) are left (resp. right) D_n-modules. Likewise for the quotients $D_n/\sum D_n P_i$ (resp. $D_n/\sum P_i D_n$). Note that $R_n = D_n/\sum D_n \partial_i$.*

Example 1.10. *Given a nonzero polynomial, $R_n[f^{-1}] = \mathbb{C}[x_1, \dots x_n, f^{-1}]$ is a D_n-module, where the derivatives act by differentiation of rational functions.*

Example 1.11. *Let F be any space of complex valued functions on $\mathbb{C}^n$ which is an algebra over the polynomial ring and and closed under differentiation, then it becomes a left D_n-module as above. In particular, this applies to holomorphic, C^∞ and C^∞ functions with compact support.*

Example 1.12. *The space of distributions is the topological dual of the space of C^∞ functions with compact support or test functions. This has a right module structure defined a follows. Given a distribution δ, a test function ϕ and $P \in D_n$, let $\delta P(\phi) = \delta(P\phi)$.*

The first four examples above are finitely generated. (The last example requires some thought. In the special case $f = x_1$, we see immediately that x_1^{-N} can be obtained by repeated differentiation.)

Fix a space of functions F as in example 1.11. Given a left D_n-module M, define the space of solutions by

$$Sol(M) = Hom_{D_n}(M, F)$$

To justify this terminology consider the example 1.9 above. We see immediately that there is an exact sequence

$$0 \to Sol(M) \to F \xrightarrow{\sum P_i} F^N$$

Therefore $Sol(M)$ is the space of solutions of the system $P_i(f) = 0$.

There is a symmetry between right and left modules that I will refer to as "flipping".

Lemma 1.13. *There is an involution $P \mapsto P^*$ of D_n determined by $x_i^* = x_i$ and $\partial_i^* = -\partial_i$. Given a right D_n-module M, the operation $P \cdot m = mP^*$ makes M into a left module, which I denote by $Flip^{R\to L}(M)$. This gives an equivalence between the categories of (finitely generated) left and right modules. The inverse operation will denote by $Flip^{L\to R}$.*

Suppose that M is a finitely generated D_n-module. We define *good* filtration on M to be a filtration F_pM such that

(1) The filtration $F_pM = 0$ for $p \ll 0$ and $\cup F_pM = M$.
(2) Each F_pM is a finitely generated R_n-submodule.
(3) $F_pD_n \cdot F_qM \subseteq F_{p+q}M$.

Lemma 1.14. *Every finitely generated module possess a good filtration.*

Proof. Write it as a quotient of some free module D_n^N and take the image of $(F_pD_n)^N$ □

The filtration is *not* unique, however it does lead to some well defined invariants. Given a module with good filtration, the associate graded

$$Gr(M) = \bigoplus F_pM/F_{p-1}M$$

is a finitely generated $Gr(D_n)$-module. The annihilator of $Gr(M)$ gives an ideal in $Gr(D_n) \cong R_{2n}$. The zero set of this ideal defines a subvariety $Ch(M, F) \subset \mathbb{C}^{2n}$ called the *characteristic variety* or singular support. Since $Gr(M)$ is graded with respect to the natural grading on $Gr(D_n)$, we see that, the annihilator is homogeneous. Therefore

Lemma 1.15. *$Ch(M, F)$ is invariant under the action of $t \in \mathbb{C}^*$ by $(x_i, \xi_j) \mapsto (x_i, t\xi_j)$.*

We can view this another way. Consider the Rees module $Rees(M, F) = \oplus F_pM$. This is a finitely generated module over H_n such that $Gr(M) = H_n/(q) \otimes Rees(M, F)$. So in some sense $Ch(M, F)$ is the classical limit of M as $q \to 0$.

Theorem 1.16. *$Ch(M, F)$ is independent of the filtration. Thus we can, and will, drop F from the notation.*

Example 1.17. *In the previous examples, we see that*

(1) *The annihilator of $Gr(D_n)$ is 0, so that $Ch(D_n) = \mathbb{C}^{2n}$.*
(2) *Taking $R_n = D_n/\sum D_n\partial_i$, yields $Gr(R_n) = \mathbb{C}[x_1, \dots x_n]$. Its annihilator is the ideal $(\xi_1, \dots \xi_n)$. Therefore $Ch(R_n) = \mathbb{C}^n \times 0$*
(3) *Let $M = R_1[x^{-1}]$, where we $x = x_1$. Then $1, x^{-1}$ generate M. Let*
$$F_kM = F_k \cdot 1 + F_k \cdot x^{-1} = \mathbb{C}[x]x^{-k-1}$$
This gives a good filtration. A simple computation shows that
$$Gr(M) \cong \mathbb{C}[x, \xi]/(\xi) \oplus \mathbb{C}[x, \xi]/(x)$$
So $Ch(M) = V(x\xi)$ is a union of the axes.

Theorem 1.18 (Bernstein's inequality)**.** *For any nonzero finitely generated D_n-module, we have* $\dim Ch(M) \geq n$*.*

There are a number of ways to prove this. Perhaps the most conceptual, though not the easiest, way is to deduce it from the involutivity of the annihilator [Ga]. This means that this ideal is closed under the Poisson bracket induced from the symplectic structure of $\mathbb{C}^{2n} = T^*\mathbb{C}^n$. This implies that the tangent space of any smooth point $p \in Ch(M)$ satisfies $T_p^\perp \subseteq T_p$, and the inequality follows. Note that the $\mathbb{C}^*$-action of lemma 1.15 is precisely the natural action on the fibers of the cotangent bundle.

We say that finitely generated D_n-module M is *holonomic* if $\dim Ch(M) = n$ or if $M = 0$. Thanks to Bernstein's inequality, this is equivalent to $\dim Ch(M) \leq n$. For example R_n and $R_1[x^{-1}]$ are holonomic, but D_n isn't.

Proposition 1.19. *The class of holonomic modules is closed under submodules, quotients and extensions. Therefore the full subcategory of holonomic modules is Abelian.*

Proof. One checks that $Ch(M_2) = Ch(M_1) \cup Ch(M_3)$ for any exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$. □

From the symplectic viewpoint, holonomic modules are precisely the ones with Lagrangian characteristic varieties. There is also a homological characterization of such modules.

Theorem 1.20. *A finitely generated D_n-module is holonomic if and only if $Ext^i(M, D_n) = 0$ for $i \neq n$. If M is holonomic, then the module $Ext^n(M, D_n)$ is a finitely generated holonomic right D_n-module. The contravariant functor $M \mapsto Flip^{R\to L} Ext^n(M, D_n)$ is an involution on the category of holonomic modules.*

Corollary 1.21. *Holonomic modules are artinian (which means the descending chain condition holds).*

Sketch. Any descending chain in M gets flipped around to an ascending chain in $N = Ext^n(M, D_n)$. D_n is known to be right (and left) noetherian, so the same goes for N. □

It will follow that holonomic modules can built up from simple holonomic modules.

1.22. **Inverse and direct image.** Suppose that $X = \mathbb{C}^n$ with coordinates x_i and $Y = \mathbb{C}^m$ with coordinates y_j. Consider a map $F : X \to Y$ given by

$$F(x_1, \dots x_n) = (F_1(x_1, \dots x_n), \dots)$$

where the F_i are polynomials. Let $O_X = \mathbb{C}[x_1, \dots x_n]$, and $O_Y = \mathbb{C}[y_1, \dots y_n]$, and let D_X and D_Y denote the corresponding Weyl algebras. (To avoid confusion, I will label the derivatives ∂_{x_i} etc.) Then F determines an algebra homomorphism

$$O_Y \to O_X$$

$$f \mapsto f \circ F$$

Given a left D_Y-module M, we define a left D_X module F^*M, called the *inverse image*, as follows. First define

$$F^*M = O_X \otimes_{O_Y} M$$

as an O_X-module. We now define an action of the derivatives by the chain rule

$$\partial_{x_i}(f\otimes m) = \frac{\partial f}{\partial x_i}\otimes m + \sum_j f\frac{\partial F_j}{\partial x_i}\otimes \partial_{y_j} m$$

Lemma 1.23. *This formula determines a D_X-module structure on F^*M*

Example 1.24. *Let $X = \mathbb{C}^n$ with coordinates $x_1, \ldots x_n$, $Y = \mathbb{C}^{n-1}$ with coordinates $x_1, \ldots x_{n-1}$. Let $p(x_1, \ldots x_n) = (x_1, \ldots x_{n-1})$. Then*

$$p^*M = \mathbb{C}[x_n]\otimes_{\mathbb{C}} M$$

Given $f(x_n)\otimes m$, x_n and ∂_n acts in the usual way through the first factor, and remaining generators of D_X act through the second.

There is a second description that is useful. Define

$$D_{X\to Y} = F^*D_Y = O_X\otimes_{O_Y} D_Y$$

This has the structure of a left D_X-module as above, as well as a right D_Y-module structure, where D_Y acts by right multiplication on itself in the above formula. These two actions commute, so they determine a so called bimodule structure. If we flip both of these actions, we get left D_Y right D_X bimodule

$$D_{Y\leftarrow X} = Flip_{D_X}^{L\to R} Flip_{D_Y}^{R\to L}(D_{X\to Y}).$$

Lemma 1.25. $f\, F^*M = D_{X\to Y}\otimes_{D_Y} M$

Proof.

$$F^*M = O_X\otimes O_Y M = (O_X\otimes_{O_Y} D_Y)\otimes_{D_Y} M = D_{X\to Y}\otimes_{D_Y} M$$

□

Given a left D_X-module N, the direct image

$$F_*N = D_{Y\leftarrow X}\otimes_{D_X} N$$

is a left D_Y-module. This operation is sometimes denoted with an integral sign to suggest the analogy with integration along the fibers.

Example 1.26. *Let $X = \mathbb{C}^n$ with coordinates $x_1 \ldots x_n$, $Y = \mathbb{C}^{n+1}$ with coordinates $x_1, \ldots x_{n+1}$ and suppose $i(x_1, \ldots) = (x_1, \ldots x_n, 0)$. We have $i_*M = M^{\mathbb{N}}$, a countable direct sum. Here $x_1, \partial_1, x_2, \ldots \partial_n$ acts as componentwise using the given D_X-module structure, x_{n+1} acts by 0, ∂_{n+1} acts as the shift operator*

$$(m_1, m_2, \ldots)\mapsto (0, m_1, m_2, \ldots)$$

Thus it is more suggestive to write

$$i_*M = \bigoplus_j \partial_{n+1}^j M$$

These operations are compatible with composition, as one would hope.

Theorem 1.27. *If $F : X\to Y$ and $G : Y\to Z$ are polynomial maps of affine spaces, then for any D_Z-module M and D_X-module M, we have*

1. $(G\circ F)^*M \cong F^*G^*M$
2. $(G\circ F)_*N \cong G_*F_*N$.

Theorem 1.28. *These operations preserve finite generation and holonomicity.*

Since any map can be factored as an embedding followed by a projection, it suffices by the theorem 1.27 to check these two cases. This theorem provides many additional examples of holonomic modules.

1.29. **Differential operators on affine varieties.** Let X be a nonsingular affine variety over $\mathbb{C}$. This is a complex manifold which can be described as the set of solutions to a system of polynomial equations in some $\mathbb{C}^n$. We write $R(X)$ for ring regular (= polynomial) functions on X. This is a finitely generated commutative algebra. A differential operator of order $\leq k$ on X is a $\mathbb{C}$-linear endomorphism T of $R(X)$ such that

$$[\ldots[[T, f_0], f_1] \ldots f_k] = 0$$

for all $f_i \in R(X)$. Let $\mathrm{Diff}_k(X)$ denote the space of these operators. We define

$$D_X = \bigcup \mathrm{Diff}_k(X)$$

Lemma 1.30. *D_X becomes a ring under composition such that $\mathit{Diff}_k \mathit{Diff}_m \subset \mathit{Diff}_{k+m}$.*

We note the following characterization (c.f. [K2, lemma 1.7]) which sometimes useful.

Proposition 1.31. *D_X is a quotient of the univeral enveloping algebra of the Lie algebra of vector fields $\mathit{Der}_{\mathbb{C}}(R(X))$ by the relations $[\xi, f] = \xi(f)$ for all $\xi \in \mathit{Der}_{\mathbb{C}}(R(X))$ and $f \in R(X)$.*

When $X = \mathbb{C}^n$, $(D_X, \mathrm{Diff}_\bullet) = (D_n, F_\bullet)$. Everything that we have done so far generalizes to the setting of affine varieties. For example

Theorem 1.32. *The associated graded with respect to $\mathit{Diff}_\bullet$ is isomorphic to the ring of regular functions on the cotangent bundle T^*X.*

We can define left/right D-modules as before. All of the previous examples generalize. We give a new example.

Example 1.33. *Let $f \in R_n$, and let X be complement of the zero set of f in $\mathbb{C}^n$. This is an affine variety with coordinate ring $R = R_n[\frac{1}{f}]$. Let $A = \sum A_i dx_i$ be an $r \times r$ matrix of 1-forms with coefficients in R satisfying the integrability condition $[A_i, A_j] = 0$. Then $M = R^r$ carries a left D_X-module structure with*

$$\partial_i v = \frac{\partial v}{\partial x_i} + A_i v;\ v \in M$$

Note that this construction is equivalent to defining an integrable connection on M. There are nontrivial examples only when X is non-simply connected, and in particular none unless $f \neq 1$.

The "flipping" operation for affine varieties is more subtle than before. Let ω_X denote the canonical module or equivalently the module of algebraic n-forms, where $n = \dim X$. This has right D_X module structure dual to left module structure on $R(X)$. Heuristically, this can be undertood by the equation

$$\int_X (\alpha P) f = \int_X \alpha (Pf)$$

where α is an n-form, P a differential operator, and f a function and X is replaced by a compact manifold. A rigorous definition can be given via the Lie derivative L. Given a vector field ξ and an element of $\alpha \in \omega_X$,

$$L_\xi \omega(\xi_1, \dots \xi_n) = \xi(\omega(\xi_1, \dots \xi_n)) + \sum \omega(\xi_1, \dots [\xi, \xi_i], \dots \xi_n)$$

Then $\omega \cdot \xi = -L_\xi \alpha$ can be shown to extend to a right action of the whole ring D_X with the help of proposition 1.31. Under this action, the difference $(\alpha P)f - \alpha(Pf)$ can be shown to be exact, and so above integral formula would follow.

Lemma 1.34. *If M is a left D_X-module, then $Flip^{L\to R}(M) = \omega_X \otimes_{R(X)} M$ carries a natural right D_X-module structure. This operation induces an equivalence between the categories of left and right D_X-modules; its inverse is $Flip^{R\to L}(N) = \omega_X^{-1} \otimes N$.*

Note that $\omega_{\mathbb{C}^n} \cong R_n$, which was why we could ignore it.

The notions of characteristic variety and holonomocity can be defined as before. The characteristic variety of example 1.33 is X embedded in T^*X as the zero section. Therefore it is holonomic.

Given a morphism of affine varieties $F : X \to Y$, we can define bimodules $D_{X\to Y}$, $D_{Y\leftarrow X}$, and inverse and direct images as before.

1.35. **Non-affine varieties.** Now we want to generalize to the case where X is a nonsingular non-affine variety, for example projective space $\mathbb{P}^n$. First, recall that in its modern formulation a variety consists of a space X with a Zariski topology and a sheaf of commutative rings $\mathcal{O}_X$, such that for any open set $\mathcal{O}_X(U)$ is the space of regular functions [Ha]. By definition, X possesses an open covering by affine varieties. Our first task is to extend D_X to this world:

Lemma 1.36. *There exists a unique sheaf of noncommutative rings D_X on X such that for any affine open U, $D_X(U)$ is the ring of differential operators on U.*

We can define a filtration by subsheaves $F_pD_X \subset D_X$ as above. The previous result globalizes easily to:

Theorem 1.37. *The associated graded is isomorphic to $\pi_*\mathcal{O}_{T^*X}$ where $\pi : T^*X \to X$ is the cotangent bundle.*

A left or right D_X-module is sheaf of left or right modules over D_X. For example, $\mathcal{O}_X$ (resp. ω_X) has a natural left (resp. right) D_X-module structure. D_X has both. We have an analogue of lemma 1.34 in this setting, so we can always switch from right to left.

We will be primarily interested in the modules which are coherent (i.e. locally finitely generated) over D_X. The notion of good filtration for a D_X-module M can be extended to this setting. The associated graded $Gr(M)$ defines a sheaf over the cotangent bundle, and the characteristic variety $Ch(M)$ is its support. This depends only on M as before and is $\mathbb{C}^*$-invariant. We have Bernstein's inequality $\dim Ch(M) \geq \dim X$, and M is holonomic if equality holds. We again have:

Proposition 1.38. *The full subcategory of holomonic modules is an artinian Abelian category.*

Given a morphism of varieties $F : X \to Y$, we define

$$D_{X\to Y} = \mathcal{O}_X \otimes_{F^{-1}\mathcal{O}_Y} F^{-1}D_Y$$

where F^{-1} is the inverse image in the category of sheaves [Ha]. This is a left D_X right $F^{-1}D_Y$ bimodule. We define a right D_X left $F^{-1}D_Y$ bimodule by flipping both actions:

$$D_{Y\leftarrow X} = Flip^{L\to R}_{D_X} Flip^{R\to L}_{F^{-1}D_Y}(D_{X\to Y})$$

As an O_X-module, it this is isomorphic to $\omega_X \otimes D_{X\to Y} \otimes F^*\omega_Y^{-1}$.

Given a left D_Y-module M, we can define the naive[1] inverse image as the D_X-module:

$$F_n^* M = D_{X\to Y} \otimes_{F^{-1}D_Y} F^{-1}M$$

For a left D_X-module N, the naive direct image as the D_Y-module

$$F_*^n N = F_*(D_{Y\leftarrow X} \otimes_{D_X} N)$$

where F_* on the right is the sheaf theoretic direct image.

The above definitions proceed in complete analogy with the affine case. The bad news is that the naive direct image is somewhat pathological. For example, the composition rule (theorem 1.27) may fail. The solution is to work in the setting of *derived categories*. Let $D^b(D_X)$ denote the bounded derived category of quasi-coherent left D_X-modules. The objects of this category are bounded (i.e. finite) complexes of left D_X-modules. Two objects are isomorphic in $D^b(D_X)$ if and only if they are quasi-isomorphic in the usual sense i.e. possess isomorphic cohomology sheaves. More formally, $D^b(D_X)$ is constructed in two steps: first one passes to the homotopy category $K^b(D_X)$ where morphisms are homotopy classes of maps between complexes, then the quasi-isomorphisms are inverted by a procedure analogous to localization in commutative algebra. The details can be found, for example, in [Bo, GM]. The categories $K^b(D_X)$ and $D^b(D_X)$ are not abelian, so that exact sequences are not meaningful inside them. However, these categories are *triangulated*, which means that they are equipped with a collection of diagrams called distinguished triangles, and this provides a reasonable substitute. The functors $F_*, \otimes, \ldots$ extend to derived functors $\mathbb{R}F_*, \otimes^{\mathbb{L}}, \ldots$ between these derived categories. In practice, this involves replacing a given complex by an appropriate (injective, flat...) resolution, before applying the functor. Taking cohomology sheaves, yields functors $\mathcal{H}^i : D^b(D_X) \to M(D_X)$ to the category of D_X-modules.

We can define the inverse image

$$F^* : D^b(D_Y) \to D^b(D_X)$$

by

$$F^* M^\bullet = D_{X\to Y} \otimes^{\mathbb{L}}_{F^{-1}D_Y} F^{-1}M^\bullet$$

and the direct image $F_* : D^b(D_X) \to D^b(D_Y)$ by

$$F_* N^\bullet = \mathbb{R}F_*(D_{Y\leftarrow X} \otimes^{\mathbb{L}}_{D_X} N^\bullet)$$

These behave well under composition. At the end of the day, we can compose these operations with $\mathcal{H}^i$ to get actual D-modules.

[1] This is nonstandard terminology.

1.39. **Connections.** Let E be a vector bundle on a nonsingular variety X, i.e. a locally free O_X-module. A D_X-module structure on E is the same thing as an integrable connection on E, which is given locally as in example 1.33. Globally, this is a $\mathbb{C}$-linear map from the tangent sheaf

$$\nabla : \mathcal{T}_X \to End(E)$$

such that $\nabla(v)$ is a derivation and such that ∇ preserves Lie bracket

$$\nabla([e_1, e_2]) = [\nabla(e_1), \nabla(e_2)]$$

Equivalently, it is given by a $\mathbb{C}$-linear map

$$\nabla : E \to \Omega^1_X \otimes E$$

satisfying the Leibnitz rule and the having curvature $\nabla^2 = 0$.

From, the local description, it is easy to see that the characteristic variety of an integrable connection is the zero section of T^*X. Thus it is holonomic. Conversely,

Proposition 1.40. *M is a vector bundle with integrable connection if and only if its characteristic variety is the zero section of T^*X*

Corollary 1.41. *If M is a holonomic module, there exists an open dense set $U \subseteq X$ such that $M|_U$ is an integrable connection.*

Proof. We can assume that the support of M is X, otherwise the statement is trivially true. Then the map $Ch(M) \to X$ is generically finite, and therefore finite over some open $U \subset X$. Since $Ch(M) \cap T^*U$ is $\mathbb{C}^*$ invariant, it must be the zero section. □

Given a morphism $F : X \to Y$ and an integrable connection (E, ∇) on Y. The pullback of the associated D_Y-module coincides with the pullback F^*E in the category of O-modules with its induced connection. If (E', ∇') is an integrable connection on X, then the pushforward of the associated D_X-module does not come from a connection in general. However, there is one important case where it does, see section 1.56.

We finally discuss the notion of *regular singularities* which is a growth condition at infinity. The classical condition is the following.

Example 1.42. *Let A be an $r \times r$ matrix of rational 1-forms on $\mathbb{P}^1$. Let U be the complement of the poles in $\mathbb{P}^1$ of the entries of A, and let $j : U \to \mathbb{P}^1$ the inclusion. Then we can define a D_U-module structure on $M = \mathcal{O}_U^r$ by*

$$\partial v = \frac{dv}{dx} + Av$$

M is holomonic. The D_X-module j_*M has regular singularities if and if the differential equation $\partial v = 0$ has regular singularities in the classical sense; this is the case if the poles of A are simple.

In general, we have the following extension due to Deligne. A vector bundle (E, ∇) with a connection on a smooth variety X has regular singularities if there exists a nonsingular compactification $\bar{X}$, with $D = \bar{X} - X$ a divisor with normal crossings, such that (E, ∇) extends to a vector bundle with a map

$$\bar{\nabla} : \bar{E} \to \Omega^1_X(\log D) \otimes \bar{E}$$

The notion of regular singularities can be extended to arbitrary holonomic D_X-modules. If M is a simple holonomic module with support Z, then $M|_Z$ is generically given by an integrable connection as above. Say that M has regular singularities if this connection is regular. In general, M has regular singularities if each of its simple subquotients have regular singularities. This notion behaves well with respect to the operations defined earlier. See [Be, Bo, K2] for details.

1.43. **Riemann-Hilbert correspondence.** In the 19th century Riemann completely analyzed the hypergeometric equation in terms of its monodromy. Hilbert, in his 21st problem, proposed that a similar analysis should be carried out for more general differential equations. Here I want to explain a very nice interpretation and solution in D-module language due to Kashiwara-Kawai and Mebkhout.

Fix a smooth variety X over $\mathbb{C}$. We can treat X as a complex manifold, and we denote this by X^{an}. Most algebraic objects give rise to corresponding analytic ones, usually marked by "an". In particular, $D_{X^{\mathrm{an}}}$-module is the sheaf of holomorphic differential operators. Any D_X-module gives rise to a $D_{X^{\mathrm{an}}}$-module.

Let $\Omega^p_{X^{\mathrm{an}}}$ denote the sheaf of holomorphic p-forms on X^{an}. Recall that we have a complex, $\Omega^\bullet_{X^{\mathrm{an}}}$, called the de Rham complex, which is quasi-isomorphic to the constant sheaf $\mathbb{C}_{X^{\mathrm{an}}}$. We can modify this to allow coefficients in any $D_{X^{\mathrm{an}}}$-module M:

$$DR(M)^\bullet = \Omega^\bullet_{X^{\mathrm{an}}} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} M[\dim X]$$

(The symbol $[n]$ mean shift the complex n places to the left). The differential is given in local coordinates by

$$d(dx_{i_1} \wedge \ldots \wedge dx_{i_p} \otimes m) = \sum_j dx_j \wedge dx_{i_1} \wedge \ldots dx_{i_p} \otimes \partial_j m$$

We can define a complex

$$\ldots D_X \otimes_{O_X} \wedge^2 \mathcal{T}_X \to D_X \otimes_{O_X} \mathcal{T}_X \to D_X$$

with differentials dual to $DR(D_X)^\bullet$ under the identification

$$Hom_{\text{right-}D_X\text{-mod}} Hom(\Omega^p_X \otimes_{O_X} D_X, D_X) \cong D_X \otimes_{O_X} \wedge^p T_X$$

The complex $D_{X^{\mathrm{an}}} \otimes_{O_{X^{\mathrm{an}}}} \wedge^\bullet \mathcal{T}_{X^{\mathrm{an}}}$ gives a locally free resolution of $O_{X^{\mathrm{an}}}$. This comes down to the fact that it becomes a Koszul complex after taking the associated graded with respect to F. Therefore

$$DR(M)^\bullet \cong Hom(D_{X^{\mathrm{an}}} \otimes_{O_{X^{\mathrm{an}}}} \wedge^\bullet \mathcal{T}_{X^{\mathrm{an}}}, M) \cong \mathbb{R}\mathcal{H}om(O_{X^{\mathrm{an}}}, M)$$

We can extend the definition of DR to the derived category $D^b(D_X)$ by using the last formula.

We can now give classical version of the Riemann-Hilbert correspondence.

Proposition 1.44. *If E is a holomorphic vector bundle with an integrable connection ∇, $DR(E)[-\dim X]$ is a locally constant sheaf of finite dimensional $\mathbb{C}$-vector spaces. The functor $E \mapsto DR(E)[-\dim X]$ induces an equivalence of categories between these categories.*

Sketch.

$$DR(E)[-\dim X] = E \xrightarrow{\nabla} \Omega^1_{X^{\mathrm{an}}} \otimes E \xrightarrow{\nabla} \ldots$$

gives a resolution of $ker\nabla$, which is locally constant. Conversely, given a locally constant sheaf L, $O_{X^{\mathrm{an}}} \otimes L$ can be equipped with an integrable connection such L is the kernel. $\square$

By imposing regularity assumptions, Deligne was able to make this correspondence algebraic [De2]. The point is that regularity ensures that the holomorphic data extends to a compactification, where GAGA applies. For general D-modules, we impose holonomicity as well. $DR(M)$ will no longer be a locally constant sheaf in general, but rather a complex with constructible cohomology. Recall that a $\mathbb{C}_{X^{\mathrm{an}}}$-module L is *constructible* if there exists an algebraic stratification of X such that the restrictions L to the strata are locally constant with finite dimensional stalks.

Theorem 1.45. *The de Rham functor DR induces an equivalence of categories between the subcategory $D_{rh}(D_X) \subset D^b(D_X)$ of complexes with regular holonomic cohomology and the subcategory $D^b_{constr}(\mathbb{C}_{X^{an}}) \subset D^b(\mathbb{C}_{X^{an}})$ of complexes with constructible cohomolgy. Moreover the inverse and direct images constructions are compatible under this correspondence.*

There is one more aspect of this, which is worth noting. The duality $M \mapsto Flip^{R\to L} Ext^n(M, D_n)$ constructed earlier can be generalized naturally to this setting to

$$M \mapsto Flip^{R\to L}\mathbb{R}\mathcal{H}om(M, D_X).$$

This corresponds to the Verdier dual

$$F \mapsto D(F) = \mathbb{R}\mathcal{H}om(F, \mathbb{C}_{X^{\mathrm{an}}}[2\dim X])$$

on the constructible derived category. This operation arises in the statement Poincaré-Verdier duality:

Theorem 1.46. *If $L \in D^b_{constr}(\mathbb{C}_{X^{an}})$, then $H^i(X, L) \cong H_c^{-i}(X, D(L))^*$*

When X is nonsingular and $L = \mathbb{C}$, we have $D(L) \cong \mathbb{C}[2\dim X]$, so this reduces to ordinary Poincaré duality.

1.47. **Perverse Sheaves.** Let X be nonsingular. The category of of regular holonomic modules sits in the triangulated category $D^b(D_X)$ as an Abelian subcategory. Its image under DR is the Abelian category of complex *perverse sheaves* [BBD]. In spite of the name, these objects are neither perverse nor sheaves, but rather a class of well behaved elements of $D^b(\mathbb{C}_{X^{\mathrm{an}}})$.

Example 1.48. $DR(\mathcal{O}_X) = \mathbb{C}_X[\dim X]$ *is perverse.*

Example 1.49. *Suppose that X is complete (e.g. projective). Suppose that V is a vector bundle with an integrable connection $\nabla : V \to \Omega^1_X(\log D) \otimes V$ with logarithmic singularities along a normal crossing divisor. Let $U = X - D$ and $j : U \to X$ be the inclusion. Then $L = \ker \nabla|_{j^*V}$ is a local system on U. Then we have*

$$DR(j_*j^*V) = \mathbb{R}j_*(L)[\dim X]$$

is perverse.

Perverse sheaves can be characterized by purely sheaf theoretic methods:

Theorem 1.50. *$F \in D^b_{constr}(\mathbb{C}_{X^{an}})$ is perverse if and only if*

(1) *For all j, $\dim supp\,\mathcal{H}^j(F) \leq -j$.*
(2) *These inequalities also hold for the Verdier dual $D(F)$*

Note that these conditions work perfectly well with other coefficients, such as $\mathbb{Q}$, to define full subcategories $Perv(\mathbb{Q}_{X^{\mathrm{an}}}) \subset D^b_{constr}(\mathbb{Q}_{X^{\mathrm{an}}})$.

Example 1.51. *If $j : U \to X$ is as in the previous example, then for any local system of $\mathbb{Q}$ vector spaces L, the sheaves $\mathbb{R}j_*L[\dim X]$ and $j_!L[\dim X]$ are perverse. The first condition can be checked directly. For the second, observse that $D(\mathbb{R}j_*L[\dim X]) = j_!L^\vee[\dim X]$ and likewise of the dual of $j_!L[\dim X]$.*

Perverse sheaves have another source, independent of D-modules. In the late 70's Goresky and Macpherson introduced intersection homology by a geometric construction by placing restrictions how chains met the singular set in terms of a function refered to as the perversity. Their motivation was to find a theory which behaved like ordinary homology for nonsingular spaces in general; for example, by satisfying Poincaré duality. When their constructions were recast in sheaf theoretic language [Br, GoM], they provided basic examples of perverse sheaves.

Example 1.52. *Suppose that $Z \subset X$ is a possible singular subvariety. Then the complex $IC_Z(\mathbb{Q})$ computing the rational intersection cohomology of Z is (after a suitable shift and extension to X) a perverse sheaf on X. This is more generally true for the complex $IC_Z(L)$ computing interesection cohomology of Z with coefficients in a locally constant sheaf defined on a Zariski open $U \subset Z$. In the notation of* [BBD]*, this would be denoted by $i_*j_{!*}L[\dim Z]$, where $j : U \to Z$ and $i : Z \to X$ are the inclusions.*

It turns out that $IC_Z(\mathbb{Q})$ is self dual under Verdier duality, and in general that $D(IC_Z(L)) \cong IC_Z(L^*)$. This implies Poincaré duality for intersection cohomology.

In typical cases[2] of example 1.49, if $L = \ker \nabla|_{j^*V}$ then $\mathbb{R}j_*L$ is quasi-isomorphic to the log complex $(\Omega_X(\log D) \otimes V, \nabla)$. Then $IC_X(L)$ can be realized by an explicit subcomplex of the log complex. It can also be realized, in many cases, by a complex of C^∞ forms on $X - D$ with L^2 growth conditions.

Theorem 1.53. *The category of perverse sheaves (over $\mathbb{Q}, \mathbb{C}, \dots$) is Artinian, and the simple objects are as in example 1.52 with Z and L irreducible.*

Corollary 1.54. *Simple perverse sheaves are generically local systems on their support.*

The last results shouldn't come as a surprise, since as we have seen previously, regular holomonomic D-modules are generically given by integrable connections.

Example 1.55. *When X is a smooth curve, this can be made very explicit. Simple perverse sheaves are either sky scraper sheaves $\mathbb{Q}_x$ with $x \in X$, or sheaves of the form $j_*L[1]$, where L is an irreducible local system on a Zariski open set $j : U \to X$.*

There is a functor ${}^p\mathcal{H}^i : D^b_{constr}(\mathbb{Q}_{X^{an}}) \to Perv(\mathbb{Q}_{X^{an}})$, which corresponds to the operation $\mathcal{H}^i : D^b_{rh}(D_X) \to M_{rh}(D_X)$ under Riemann-Hilbert.

1.56. **Gauss-Manin connections.** Suppose that $f : X \to Y$ is a smooth and proper morphism of relative dimension n. Let $\Omega^\bullet_{X/Y}$ be the sheaf of relative differentials. Then for any D_X-module we have a relative de Rham complex

$$DR_{X/Y}(M) = \Omega^\bullet_{X/Y} \otimes M$$

$DR_{X/Y}(D_X)$ gives a resolution of $D_{Y\leftarrow X}$, with augmentation

$$\Omega^n_{X/Y} \otimes D_X \cong \omega_X \otimes f^{-1}\omega_Y \otimes D_X \to D_{Y\leftarrow X}$$

[2]The precise meaning of "typical" here is that the residues of the connection should have no positive integer eigenvalues

Then the direct image

$$\mathbb{R}f_*(D_{Y\leftarrow X}\otimes^{\mathbb{L}}_{D_X} M) = \mathbb{R}f_*(DR_{X/Y}(D_X)\otimes^{\mathbb{L}}_{D_X} M) = \mathbb{R}f_*(DR_{X/Y}(M)) \tag{1}$$

When $M = O_X$, the ith cohomology sheaf of the direct image

$$R^if_*DR_{X/Y}(O_X) = R^if_*\Omega^\bullet_{X/Y}$$

is locally free with an integrable connection called the Gauss-Manin connection. Under Riemann-Hilbert, Gauss-Manin corresponds to the locally constant sheaf $R^if_*\mathbb{C}$. This connection was constructed in pre-D-module language as follows. Assume $\dim Y = 1$ for simplicity, then the connection is the connecting map

$$R^if_*\Omega^\bullet_{X/Y} \to \Omega^1_Y \otimes R^if_*\Omega^\bullet_{X/Y}$$

associated to the sequence

$$0 \to \Omega^1_Y \otimes \Omega^\bullet_{X/Y}[-1] \to \Omega^\bullet_X \to \Omega^\bullet_{X/Y} \to 0$$

When f is not smooth, then by resolution of singularities, the singular fibers E of f can be assumed to have normal crossings. Then the above discussion can be extended to the log complexes, resulting in a map

$$R^if_*\Omega^\bullet_{X/Y}(\log E) \to \Omega^1_Y(\log f(E)) \otimes R^if_*\Omega^\bullet_{X/Y}(\log E)$$

This gives a proof (due to Katz) that the Gauss-Manin connection has regular singularities.

2. Vanishing cycles

2.1. **Vanishing cycles.** Vanishing cycle sheaves and their corresponding D-modules form the basis for Saito's constructions described later. I will start with the classical picture. Suppose that $f : X \to \mathbb{C}$ is a morphism from a nonsingular variety. The fiber $X_0 = f^{-1}(0)$ may be singular, but the nearby fibers X_t, $0 < |t| < \epsilon \ll 1$ are not. The premiage of the ϵ-disk $f^{-1}D_\epsilon$ retracts onto X_0, and $f^{-1}(D_\epsilon - \{0\}) \to D_\epsilon - \{0\}$ is a fiber bundle. Thus we have a monodromy action by the (counterclockwise) generator $T \in \pi_1(\mathbb{C}^*, t)$ on $H^i(X_t)$. (From now on, I will tend to treat algebraic varieties as an analytic spaces, and will no longer be conscientious about making a distinction.) The image of the restriction map

$$H^i(X_0) = H^i(f^{-1}D_\epsilon) \to H^i(X_t),$$

lies in the kernel of $T - 1$. The restriction is dual to the map in homology which is induced by the (nonholomorphic) collapsing map of X_t onto X_0; the cycles which die in the process are the vanishing cycles.

Let me reformulate things in a more abstract way following [SGA7]. The *nearby cycle* functor applied to $F \in D^b(X)$ is

$$\mathbb{R}\Psi F = i^*\mathbb{R}p_*p^*F,$$

where $\tilde{\mathbb{C}}^*$ is the universal cover of $\mathbb{C}^* = \mathbb{C} - \{0\}$, and $p : \tilde{\mathbb{C}}^* \times_{\mathbb{C}} X \to X$, $i : X_0 = f^{-1}(0) \to X$ are the natural maps. The *vanishing cycle* functor $\mathbb{R}\Phi F$ is the mapping cone of the adjunction morphism $i^*F \to \mathbb{R}\Psi F$, and hence it fits into a distinguished triangle

$$i^*F \to \mathbb{R}\Psi F \xrightarrow{can} \mathbb{R}\Phi F \to i^*F[1]$$

Both $\mathbb{R}\Psi F$ and $\mathbb{R}\Phi F$ are somewhat loosely refered to as sheaves of vanishing cycles. These objects possess natural monodromy actions by T. If we give i^*F the trivial T action, then the diagram with solid arrows commutes.

$$\begin{tikzcd}
i^*F \arrow[r] \arrow[d] & \mathbb{R}\psi_*F \arrow[r, "can"] \arrow[d, "T-1"] & \mathbb{R}\phi_*F \arrow[r] \arrow[d, dashed, "var"] & i^*F[1] \arrow[d] \\
0 \arrow[r] & \mathbb{R}\psi_*F \arrow[r, "="] & \mathbb{R}\psi_*F \arrow[r] & 0
\end{tikzcd}$$

Thus we can deduce a morphism var, which completes this to a morphism of triangles. In particular, $T - 1 = var \circ can$. One can also show that $can \circ var = T - 1$.

Given $p \in X_0$, let B_ϵ be an ϵ-ball in X centered at p. Then $f^{-1}(t) \cap B_\epsilon$ is the so called Milnor fiber. The stalks

$$\mathcal{H}^i(\mathbb{R}\Psi\mathbb{Q})_p = H^i(f^{-1}(t) \cap B_\epsilon, \mathbb{Q})$$

$$\mathcal{H}^i(\mathbb{R}\Phi\mathbb{Q})_p = \tilde{H}^i(f^{-1}(t) \cap B_\epsilon, \mathbb{Q})$$

give the (reduced) cohomology of the Milnor fiber. And

$$H^i(X_0, \mathbb{R}\Psi\mathbb{Q}) = H^i(f^{-1}(t), \mathbb{Q})$$

is, as the terminology suggests, the cohomology of the nearby fiber. We have a long exact sequence

$$\ldots H^i(X_0, \mathbb{Q}) \to H^i(X_t, \mathbb{Q}) \xrightarrow{can} H^i(X_0, \mathbb{R}\Phi\mathbb{Q}) \to \ldots$$

The following is a key ingredient in the whole story [BBD]:

Theorem 2.2 (Gabber). *If L is perverse, then so are $\mathbb{R}\Psi L[-1]$ and $\mathbb{R}\Phi L[-1]$.*

We set ${}^p\psi_f L = {}^p\psi L = \mathbb{R}\Psi L[-1]$ and ${}^p\phi_f L = {}^p\phi L = \mathbb{R}\Phi L[-1]$.

2.3. **Perverse Sheaves on a disk.** Let D be a disk with the standard coordinate function t, and inclusion $j : D - \{0\} = D^* \to D$. For simplicity assume $1 \in D^*$. Consider a perverse sheaf F on D which is locally constant on D^*. Then we can form the diagram

$$ {}^p\psi_t F \underset{var}{\overset{can}{\rightleftarrows}} {}^p\phi_t F $$

Note that the objects in the diagram are perverse sheaves on $\{0\}$ i.e. vector spaces. This leads to the following elementary description of the category due to Deligne and Verdier (c.f. [V, sect 4]).

Proposition 2.4. *The category of perverse sheaves on the disk D which are locally constant on D^* is equivalent to the category of quivers of the form*

$$ \psi \underset{v}{\overset{c}{\rightleftarrows}} \phi $$

i.e. finite dimensional vector spaces ϕ, ψ with maps as indicated.

To get a sense of why this is true, let us explain how to construct perverse sheaves associated to certain quivers. We see immediately that

$$ 0 \rightleftarrows V $$

corresponds to the sky scraper sheaf V_0.

Let L be a local system L on D^* with monodromy given by $T : L_1 \to L_1$. Then the perverse sheaf $j_*L[1]$ corresponds to

$$ L_1 \underset{v}{\overset{c}{\rightleftarrows}} \frac{L_1}{\ker(T-I)} $$

where c is the projection, v is induced by $T - I$. Thus a quiver

$$ \psi \underset{v}{\overset{c}{\rightleftarrows}} \phi $$

with c surjective arises from $j_*L[1]$, where $L_1 = \psi$ with $T = I + v \circ c$.

It is easy to classify the simple quivers and see that they are covered by these cases. There are three types:

(P_1) $0 \rightleftarrows \mathbb{C}$ which corresponds to $\mathbb{C}_0$.

(P_2) $\mathbb{C} \rightleftarrows 0$ which corresponds to $\mathbb{C}_D[1]$.

($P_{3,\lambda}$) $\mathbb{C} \underset{\lambda-1}{\overset{=}{\rightleftarrows}} \mathbb{C}$ with $\lambda \neq 0$. This corresponds to $j_*L[1]$, where L is a rank one local system with monodromy $\lambda \neq 1$

The examples considered above are not just perverse but in fact intersection cohomology complexes. In general, they can be characterized by:

Lemma 2.5. *A quiver*

$$ \psi \underset{v}{\overset{c}{\rightleftarrows}} \phi $$

corresponds to a direct sum of intersection cohomology complexes if and only if

$$ \phi = image(c) \oplus ker(v). $$

2.6. **Kashiwara-Malgrange filtration.** Some of the original motivation for the ideas in this section came from the study of Bernstein-Sato polynomials or b-functions. However, this connection would take us took far afield. Instead we start with the question: Under Riemann-Hilbert, what is the D-module analogue of vanishing cycles? I want to start with a prototype, which should make the remainder easier to swallow.

Example 2.7. *Let $Y = \mathbb{C}$ with coordinate t. Fix a rational number $r \in (-1, 0)$, and let $M = O_{\mathbb{C}}[t^{-1}]t^r$ with ∂_t acting on the left in the usual way. For each $\alpha \in \mathbb{Q}$, define $V_\alpha M \subset M$ to be the $\mathbb{C}$-span of $\{t^{n+r} \mid n \in \mathbb{Z}, n + r \geq -\alpha\}$. The following properties are easy to check:*

1. *The filtration is exhaustive and left continuous: $\cup V_\alpha M = M$ and $V_{\alpha+\epsilon}M = V_\alpha M$ for $0 < \epsilon \ll 1$.*
2. *Each $V_\alpha M$ is stable under $t^i\partial_t^j$ if $i \geq j$.*
3. *$\partial_t V_\alpha M \subseteq V_{\alpha+1}M$, and $tV_\alpha M \subseteq V_{\alpha-1}M$.*
4. *The associated graded*
$$Gr^V_\alpha M = V_\alpha M/V_{\alpha-\epsilon}M = \begin{cases} \mathbb{C}t^{-\alpha} & \textit{if } \alpha \in r + \mathbb{Z} \\ 0 & \textit{otherwise} \end{cases}$$
is an eigenspace of $t\partial_t$ with eigenvalue $-\alpha$.

(4) implies that the set of indices where $V_\alpha M$ jumps is $r + \mathbb{Z}$ and hence discrete. Such a filtration is called discrete.

Let $f : X \to \mathbb{C}$ be a holomorphic function, and let $i : X \to X \times \mathbb{C} = Y$ be the inclusion of the graph. Let t be the coordinate on $\mathbb{C}$, and let

$$V_\alpha D_Y = D_{X\times\{0\}}\text{-module generated by } \{t^i\partial_t^j \mid i - j \geq -[\alpha]\}$$

for $\alpha \in \mathbb{Q}$. In particular, $t \in V_{-1}D_Y$ and $\partial \in V_1 D_Y$. Note that $V_0 D_Y \subset D_Y$ is the subring preserving the ideal (t).

Let M be a regular holonomic D_X-module. It is called called quasiunipotent along $X_0 = f^{-1}(0)$ if ${}^p\psi_f(DR(M))$ is quasiunipotent with respect to the action of $T \in \pi_1(\mathbb{C}^*)$. Set $\tilde{M} = i_*M$. Note that in the previous example, instead of working in $\mathbb{C} \times \mathbb{C}$, we were projecting onto the second $\mathbb{C}$, since no information is lost in this case.

Theorem 2.8 (Kashiwara, Malgrange)**.** *There exists at most one filtration $V_\bullet\tilde{M}$ on $\tilde{M}$ indexed by $\mathbb{Q}$, such that*

1. *The filtration is exhaustive, discrete and left continuous.*
2. *Each $V_\alpha\tilde{M}$ is a coherent V_0D_Y-submodule.*
3. *$\partial_t V_\alpha\tilde{M} \subseteq V_{\alpha+1}\tilde{M}$, and $tV_\alpha\tilde{M} \subseteq V_{\alpha-1}\tilde{M}$ with equality for $\alpha < 0$.*
4. *$Gr^V_\alpha\tilde{M}$ is a generalized eigenspace of $t\partial_t$ with eigenvalue $-\alpha$.*

If M is quasiunipotent along X_0, then $V_\bullet\tilde{M}$ exists.

Given a perverse sheaf L and $\lambda \in \mathbb{C}$, let ${}^p\psi_{f,\lambda}L$ and ${}^p\phi_{f,\lambda}L$ be the generalized λ-eigensheaves of ${}^p\psi_f L$ and ${}^p\phi_f L$ under the T-action. Note that $N' = T - \lambda$ gives a nilpotent endomorphism of these sheaves. For various reasons, it is better to work the logarithm $N = \log(I + N') = N' - \frac{1}{2}(N')^2 + \ldots$ which is again nilpotent. Saito [S1] has defined a modification Var of var which plays an analogous role for N.

Example 2.9. *Continuing with example 2.7, note that M is a simple $D_{\mathbb{C}}$-module, so it should correspond to a simple perverse sheaf L. I claim that (after restricting to a disk) $L = P_{3,\lambda}$, for $\lambda = \exp(2\pi i r)$, in the above classification. To see this, set $t = \exp(2\pi i\tau) \in \mathbb{C}^*$. Then the monondromy $\tau \mapsto \tau + 1$ is given by $t^r \mapsto \lambda t^r$ as required. In this case,*

$$^p\phi_t L = {}^p\phi_t L = {}^p\phi_{t,\lambda} L = {}^p\psi_{t,\lambda} L$$

Theorem 2.10 (Kashiwara, Malgrange)**.** *Suppose that $L = DR(M)$. Let $\alpha \in \mathbb{Q}$ and $\lambda = e^{2\pi i\alpha}$. Then*

$$DR(Gr^V_\alpha \tilde{M}) = \begin{cases} {}^p\psi_{f,\lambda} L & \text{if } \alpha \in [-1,0) \\ {}^p\phi_{f,\lambda} L & \text{if } \alpha \in (-1,0] \end{cases}$$

The endomorphisms $t\partial_t - \alpha, \partial_t, t$ on the left corresponds to N, can, Var respectively, on the right.

3. Hodge modules

3.1. **Hodge theory background.** A (rational) pure Hodge structure of weight $m \in \mathbb{Z}$ consists of a finite dimensional vector space $H_{\mathbb{Q}}$ with a bigrading

$$H = H_{\mathbb{Q}} \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{pq}$$

satisfying $\bar{H}^{pq} = H^{qp}$. Such structures arise naturally from the cohomology of compact Kähler manifolds. For smooth projective varieties, we have further constraints namely the existence of a polarization on its cohomology. If m is even (odd), a polarization on a weight m Hodge structure H is a (anti)symmetric quadratic form Q on $H_{\mathbb{Q}}$ satisfying the Hodge-Riemann relations. Given a Hodge structure of weight m, its Hodge filtration is the decreasing filtration

$$F^p H = \bigoplus_{p' \geq p} H^{p', m-p'}$$

The decomposition can be recovered from the filtration by

$$H^{pq} = F^p \cap \bar{F}^q$$

Deligne extended Hodge theory to singular varieties. The key definition is that of a mixed Hodge structure. This consists of a bifiltered vector space (H, F, W), with (H, W) defined over $\mathbb{Q}$, such that F induces a pure Hodge structure of weight k on $Gr_k^W H$. This refines the notion of a pure Hodge structure. A pure Hodge structure of weight k can be regarded as a mixed Hodge structure such that $Gr_k^W H = H$. Mixed Hodge structures form a category in the obvious way. Morphisms are rational linear maps preserving filtrations.

Theorem 3.2 (Deligne)**.** *The singular rational cohomology of an algebraic variety carries a canonical mixed Hodge structure.*

Griffiths introduced the notion a *variation of Hodge structure* (VHS) to describe the cohomology of family of varieties $y \mapsto H^m(X_y)$, where $f : X \to Y$ is a smooth projective map. A variation of Hodge structure of weight m on a complex manifold Y consists of the following data:

(1) A locally constant sheaf L of $\mathbb{Q}$ vector spaces with finite dimensional stalks.
(2) A vector bundle with an integrable connection (E, ∇) plus an isomorphism $DR(E) \cong L \otimes \mathbb{C}[\dim Y]$.
(3) A filtration $F^\bullet$ of E by subbundles satisfying Griffiths' transversality: $\nabla(F^p) \subseteq F^{p-1}$.
(4) The data induces a pure Hodge structure of weight m on each of the stalks L_y.

A polarization is a flat pairing $Q : L \times L \to \mathbb{Q}$ inducing polarizations in the stalks. The main example is:

Example 3.3. *If $f : X \to Y$ is smooth and projective, $L = R^m f_* \mathbb{Q}$ underlies a polarizable VHS of weight m. E is the associated vector bundle with its Gauss-Manin connection.*

The key analytic fact which makes the rest of the story possible is the following. (It was originally proved by Zucker for curves).

Theorem 3.4 (Cattani-Kaplan-Schmid, Kashiwara-Kawai)**.** *Let X the complement of a divisor with normal crossings in a compact Kähler manifold. Then intersection cohomology with coefficients in a polarized VHS on X is isomorphic to L^2 cohomology for a suitable complete Kähler metric on the X.*

It follows that L^2 cohomology is finite dimensional in this case (there is no a priori reason why it should be). When combined with the Kähler identities, we get

Corollary 3.5. *Intersection cohomology with coefficients in a polarized VHS carries a natural pure Hodge structure.*

3.6. **Filtered D-modules.** The notion of a VHS is no longer adequate to describe what's going on for nonsmooth maps. Saito defined the notion Hodge modules which gives a good theory of VHS with singularities. I start by describing the basic setting next.

Fix a smooth variety X. We define the category $MF_{rh}(X)$ to consist of regular coherent D_X-modules with good filtrations and filtration preserving morphisms. Although not Abelian it is an exact category, so we can form its derived category [BBD]. Recall, that among the requirements of "goodness" is that $F_qD \cdot F_pM \subseteq F_{p+q}M$. It suffices to check this for $q = 1$. After reindexing $F^pM = F_{-p}M$, we see that this condition is just Griffiths transverality! Although I will say nothing about proofs, it is worth remarking that many technical issues are handled by passing to the Rees module $\oplus F^pM$ over the Rees algebra.

The previously defined operations can be extended to the filtered setting. Given $(M, F) \in MF_{rh}(X)$ and a proper morphism $f : X \to Y$. We define direct image explicitly. Let $n = \dim X - \dim Y$. We break the definition into cases:

(1) If f is an embedding:
$$F_p(\mathbb{R}f_*(M, F)) = f_*(\sum_k F_kD_{Y\leftarrow X} \otimes F_{p-k+n}(M[-n]))$$
where $F_kD_{Y\leftarrow X}$ is induced by the order filtration.
(2) If f is smooth then using the formulas of section 1.56
$$F_p(\mathbb{R}f_*(DR_{X/Y}(M))) = \mathbb{R}f_*(\ldots \Omega^i_{X/Y} \otimes F_{p+i}M \ldots)$$
(3) In general, factor f into a composition of the inclusion of the graph followed by a projection, and apply the previous cases.

We can filter the cohomology modules by
$$F_p(\mathcal{H}^i(\mathbb{R}f_*(M, F))) = im[\mathcal{H}^i(F_p(\mathbb{R}f_*(M, F))) \to \mathcal{H}^i(\mathbb{R}f_*(M, F))]$$
The direct image is called *strict* if the above maps are injections.

Now add a rational structure by defining the category $MF_{rh}(D_X, \mathbb{Q})$. The objects consist of

(1) A perverse sheaf L over $\mathbb{Q}$.
(2) A regular holonomic D_X-module M with an isomorphism $DR(M) \cong L\otimes\mathbb{C}$.
(3) A good filtration F on M.

Variations of Hodge structure give examples of such objects. We can define direct images for these things by combining the previous constructions. Given a morphism $f : X \to Y$ and an object $(M, F, L) \in MF_{rh}(X, \mathbb{Q})$, the direct images are defined by
$${}^pR^if_*(M, F, L) = (\mathcal{H}^i(\mathbb{R}f_*(M, F)), {}^p\mathcal{H}^i(\mathbb{R}f_*L))$$

The notation is chosen to avoid confusion with the usual direct image for constructible sheaves. Sometimes $\mathcal{H}^i \int_f$ or $R^i f_+$ are used in the literature. When $M = (\mathbb{Q}_X, O_X)$ with trivial filtration, then for any smooth projective map $f : X \to Y$, ${}^pR^i f_* M$ gives the standard example of a VHS (up to shift).

3.7. **Hodge modules on a curve.** The category $MF_{rh}(X, \mathbb{Q})$ is really too big to do Hodge theory, and Saito defines the subcategory of (polarizable) Hodge modules which provides the right setting. The definition of this subcategory is extremely delicate, and will be explained later. To ease our journey, we will give a direct describe Hodge modules on a smooth projective curve X (fixed for this section).

Given an inclusion of a point $i : \{x\} \to X$ and a polarizable pure Hodge structure $(H, F, H_{\mathbb{Q}})$, the D-module pushforward i_*H with the filtration induced by F, together with the skyscraper sheaf $H_{\mathbb{Q},x}$ defines an object of $MF_{rh}(X)$. Let's call these polarizable Hodge modules of type 0.

Given a polarizable variation of Hodge structure (E, F, L) over a Zariski open subset $j : U \to X$, we define an object of $MF_{rh}(X)$ as follows. The underlyung perverse sheaf is $j_*L[1]$ which is the intersection cohomology complex for L. We can extend (E, ∇) to a vector bundle with logarithmic connection on X in many ways. The ambiguity depends on the eigenvalues of the residues of the extension which are determined mod $\mathbb{Z}$.

Example 3.8. *Suppose that* $(E, \nabla) = (O_{D^*}, d + r\frac{dt}{t})$ *locally. Then* E *can be extended to the whole disk by the obvious way as* O_D*, but also as* $O_D t^n$ *for any* $n \in \mathbb{Z}$*. In general, writing the connection with respect to the new trivialization has the effect of translating the residue* r *by* n*. Then the multivalued function* t^{-r+n} *gives a solution to* $\nabla f = 0$*. Its monodromy is given by multiplication by* $exp(-2\pi i(r + n))$*, and this is independent of* n*.*

For every half open interval I of length 1, there is a unique extension $\bar{E}^I$ with eigenvalues in I. Let $M' = \cup \bar{E}^I \subset j_*E$. This is a D_X-module which corresponds to the perverse sheaf $\mathbb{R}j_*L[1] \otimes \mathbb{C}$. Let $M \subset M'$ be the sub D_X-module generated by $\bar{E}^{(-1,0]}$. This corresponds to what we want, namely $j_*L[1]$. We filter this by

$$F_p\bar{E}^{(-1,0]} = j_*F_pE \cap \bar{E}^{(-1,0]}$$

$$F_pM = \sum_i F_i D_X F_{p-i}\bar{E}^{(-1,0]}$$

Then $(M, F_\bullet M, j_*L[1])$ defines an object of $MF_{rh}(X)$ that we call a polarizable Hodge module of type 1. A polarizable Hodge module is a finite direct sum of objects of these two types. Let $MH(X)^{pol}$ denote the full subcategory of these.

Theorem 3.9. $MH(X)^{pol}$ *is abelian and semisimple.*

In outline, the proof can be reduced to the following observations. We claim that there are no nonzero morphisms between objects of type 0 and type 1. To see this, we can replace X by a disk and assume that x and U above correspond to 0 and D^* respectively. Then by lemma 2.4, the perverse sheaves of type 0 and 1 correspond to the quivers

$$0 \rightleftarrows \phi$$

$$\psi \underset{1-1}{\overset{onto}{\rightleftarrows}} \phi$$

and the claim follows. We are thus reduced to dealing with the types seperatedly. For type 0 (respectively 1), we immediately reduce it the corresponding statements for the categories of Hodge structures (respectively VHS), where it is standard. In essence the polarizations allow one to take orthogonal complements, and hence conclude semisimplicity.

Theorem 3.10. *If $\mathcal{M} \in MH(X)^{pol}$, then its cohomology $H^i(\mathcal{M})$ carries a Hodge structure.*

Here is the proof. Since $MH(X,n)^{pol}$ is semisimple, we can assume that $\mathcal{M}$ is simple. Then either $\mathcal{M}$ is supported at point or it is of type 1. In the first case, $H^0(\mathcal{M}) = \mathcal{M}$ is already a Hodge structure by definition, and the higher cohomologies vanish. In the second case, we appeal to theorem 3.4 or just the special case due to Zucker.

3.11. **VHS on a punctured disk.** We want to analyse Hodge modules locally. This will provide important clues for the higher dimensional case. We may as well concentrate on the only interesting case of modules of type 1, that is variations of Hodge structure. The key statements are due to Schmid [Sc].

Proposition 3.12 (Jacobson, Morosov)**.** *Fix an integer m. Let N be a nilpotent endomorphism of a finite dimensional vector space E over a field of characteristic* 0 *(or more generally an object in artinian abelian category linear over such a field). Then the there is a unique filtration*

$$0 \subset W_{m-l} \subset \dots W_m \subset \dots W_{m+l} = E$$

called the monodromy filtration on E centered at m, characterized by following properties:

(1) $N(W_k) \subset W_{k-2}$
(2) N^k *induces an isomorphism* $Gr^W_{m+k}(E) \cong Gr^W_{m-k}(E)$

Note that this result applies to the categories of perverse sheaves and holonomic modules.

Example 3.13. *If $N^2 = 0$, the filtration is simply $im(N) \subset ker(N) \subset E$*

The last part of the proposition is reminiscent of the hard Lefschetz theorem. There is an analogous decomposition into primitive parts:

Corollary 3.14.

$$Gr^W_k(E) = \bigoplus_i N^i PGr^W_{k+2i}(E)$$

where

$$PGr^W_{m+k}(E) = ker[N^{k+1} : Gr^W_{m+k}(E) \to Gr^W_{m-k-2}(E)]$$

Things can be refined a bit in the presence of a nondegenerate form.

Lemma 3.15. *If S is a nondegenerate (skew) symmetric form on E which N preserves infinitesimally ($S(Nu,v)+S(u,Nv) = 0$), then $PGr^W_{m+k}E$ carries a non-degenerate (skew) form given by $(u,v) \mapsto S(u,N^k v)$. This induces a form $Gr^W_{m+k}E$ for which the primitive decomposition is orthogonal.*

Let $(L, \mathcal{E}, F)$ be a polarized variation of Hodge structure of weight m over a punctured disk D^*; arising for example from the cohomology of family of varieties over D^*. After choosing a trivialization of $\mathcal{E}$, we can identify the fibers over $t \in D^*$ with a fixed vector space E possessing a rational lattice $E_{\mathbb{Q}}$ and a family of flags $F_t^\bullet$. Since L is locally constant, it corresponds to a representation of $\pi_1(D^*) \to Aut(E_{\mathbb{Q}})$. It is known that the (counterclockwise) generator $T \in \pi_1(D^*)$ acts quasi-unipotently. Thus after passing to a branched cover $D \to D$, we can assume tha T acts unipotently. Then $N = \log(T)$ will be nilpotent. So we get an associated mondromy filtration W as above centered at m

Theorem 3.16 (Schmid). *For an appropriate trivialization, $F_t^\bullet$ converges in a flag variety as $t \to 0$. The limit filtration $\lim F_t^\bullet$ together with W yields a mixed Hodge structure, called the limit mixed structure on E. Moreover, the polarization of the VHS induces one on the associated graded as in the above lemma.*

In geometric situations, this yields a natural mixed Hodge structure on the cohomology of the nearby fiber. (A alternative construction of this was given by Steenbrink.) The key consequence of importance here is:

Corollary 3.17. *$Gr_k^W(E)$ with the filtration induced from $\lim F_t^\bullet$ is a pure polarizable Hodge structure of weight k.*

For Hodge modules in general, a refined version of the above statement is taken as an axiom.

3.18. **Hodge modules: introduction.** In the next section, we will define the full subcategories $MH(X, n) \subset MF_{rh}(X, \mathbb{Q})$ of *Hodge modules* of weight $n \in \mathbb{Z}$ in general. Since this is rather technical, we start by explaining the main results.

Theorem 3.19 (Saito). *$MH(X, n)$ is abelian, and its objects possess strict support decompositions, i.e. that the maximal sub/quotient module with support in a given $Z \subset X$ can be split off as a direct summand. There is an abelian subcategory $MH(X, n)^{pol}$ of polarizable objects which is semisimple.*

We essenitally checked these properties for polarizable Hodge modules on curves in section 3.7. They have strict support decompositions by the way we defined them. Let $MH_Z(X, n) \subset MH(X, n)$ denote the subcategory of Hodge modules with strict support in Z, i.e. that all sub/quotient modules have support exactly Z. The main examples are provided by the following.

Theorem 3.20 (Saito). *Any weight n polarizable variation of Hodge structure $(L, \ldots)$ over an open subset of a closed subset*

$$U \xrightarrow{j} Z \xrightarrow{i} X$$

can be extended to a polarizable Hodge module in $MH_Z(X, n)^{pol} \subset MH(X, n)^{pol}$. The underlying perverse sheaf of the extension is the associated intersection cohomology complex $i_* j_{!*} L[dimU]$. All simple objects of $MH(X, n)^{pol}$ are of this form.

Finally, there is stability under direct images.

Theorem 3.21 (Saito). *Let $f : X \to Y$ be a projective morphism with relatively ample line bundle ℓ. If $\mathcal{M} = (M, F, L) \in MH(X, n)$ is polarizable, then*

$$^pR^i f_* \mathcal{M} \in MH(Y, n + i)$$

is strict. Moreover, a hard Lefschetz theorem holds:

$$\ell^j : {}^pR^{-j}f_*\mathcal{M} \cong {}^pR^j f_*\mathcal{M}(j)$$

Corollary 3.22. *Given a polarizable variation of Hodge structure defined on an open subset of X, its intersection cohomology carries a pure Hodge structure. This cohomology satisfies the Hard Lefschetz theorem.*

The last statement was originally obtained in the geometric case in [BBD]. The above results yield a Hodge theoretic proof of the decomposition theorem of [loc. cit.]

Corollary 3.23. *With assumptions of the theorem $\mathbb{R}f_*\mathbb{Q}$ decomposes into a direct sum of shifts of intersection cohomology complexes.*

3.24. **Hodge modules: conclusion.** We now give the precise definition of Hodge modules. This is given by induction on dimension of the support. This inductive process is handled via vanishing cycles. We start by explaining how to extend the construction to $MF_{rh}(X,\mathbb{Q})$. Given a morphism $f : X \to \mathbb{C}$, and a D_X-module, we introduced the Kashiwara-Malgrange filtration V on M earlier in section 2.6. Now suppose that we have a good filtration F on M. The pair (M,F) is said to be be *quasi-unipotent and regular* along $f^{-1}(0)$ if the following conditions hold:

(1) $t(F_pV_\alpha\tilde{M}) = F_pV_{\alpha-1}\tilde{M}$ for $\alpha < 1$.
(2) $\partial_t(F_pGr^V_\alpha\tilde{M}) = F_{p+1}Gr^V_{\alpha+1}\tilde{M} \cap (\partial_t Gr^V_\alpha\tilde{M})$ for $\alpha \geq 0$.

It is worth noting that the corresponding statements in the nonfiltered case come for free, once we know that V exists of course. Also the basic example of a variation of Hodge structure on the disk satisfies these conditions with respect to the identity function.

Lemma 3.25. *If $\alpha \neq 0$, then $t : Gr^V_\alpha\tilde{M} \cong Gr^V_{\alpha-1}\tilde{M}$ and $\partial_t : Gr^V_{\alpha-1}\tilde{M} \cong Gr^V_\alpha\tilde{M}$*

We extend the functors ϕ and ψ to $MF_{rh}(X,\mathbb{Q})$. Let

$$\psi_f(M,F,L) = (\bigoplus_{-1\leq\alpha<0} Gr^V_\alpha(\tilde{M}), F[1], {}^p\psi_f L)$$

$$\phi_{f,1}(M,F,L) = (Gr^V_0(\tilde{M}), F, {}^p\phi_{f,1}L)$$

Here F actually denotes the filtration induced by it on the associated graded.

The elementary definition for curves given earlier will turn to be equivalent. Define $X \mapsto MH(X,n)$ to be the smallest collection of full subcategories of $MF_{rh}(X,\mathbb{Q})$ satisfying:

(MH1) If $(M,F,L) \in MF_{rh}(X,\mathbb{Q})$ has zero dimensional support, then it lies in $MH(X,n)$ iff its stalks are Hodge structures of weight n.

(MH2) If $(M,F,L) \in MHS(X,n)$ and $f : U \to \mathbb{C}$ is a general morphism from a Zariski open $U \subseteq X$, then
   (a) $(M,F)|_U$ is quasi-unipotent and regular with respect to f.
   (b) $(M,F,L)|_U$ decomposes into a direct sum of a module supported in $f^{-1}(0)$ and a module for which no sub or quotient module is supported in $f^{-1}(0)$.
   (c) If W is the monodromy filtration of $\psi_f(M,F,L)|_U$ (with respect to the log of the unipotent part of monodromy) centered at $n-1$, then $Gr^W_i\psi_f(M,F,L)|_U \in MH(U,i)$. Likewise for $Gr^W_i\phi_{f,1}(M,F,L)$ with W centered at n.

This is a lot to absorb, so let me make few remarks about the definition.

- If $f^{-1}(0)$ is in general position with respect to $supp\,M$, the dimension of the support drops after applying the functors ϕ and ψ. Thus this *is* an inductive definition.
- The somewhat technical condition (b) ensures that Hodge modules admit strict support decompositions. The condition can be rephrased as saying that (M, F, L) splits as a sum of the image of *can* and the kernel of *var*. A refinement of lemma 2.5 shows that L will then decompose into a direct sum of intersection cohomology complexes. This is ultimately needed to be able to invoke theorem 3.4 when the time comes to construct a Hodge structure on cohomology.
- Although $MF_{rh}(X, \mathbb{Q})$ is not an abelian category, the category of compatible pairs consisting of a D-module and perverse sheaf is. Thus we do get a W filtration for $\psi_f(M, F, L)|_U$ in (c) by proposition 3.12 by first suppressing F, and then using the induced filtration.

There is a notion of polarization in this setting. Given $(M, F, L) \in MH_Z(X, n)$, a polarization is a pairing $S : L \otimes L \to \mathbb{Q}_X[2\dim X](-n)$ satisfying certain axioms. The key conditions are again inductive. When Z is a point, S should correspond to a polarization on the Hodge structure at the stalk in the usual sense. In general, given a (germ of a) function $f : Z \to \mathbb{C}$ which is not identically zero, S should induce a polarization on the nearby cycles $Gr^W_\bullet \psi_f L[-1]$ (using the same recipe as lemma 3.15). Once all the definitions are in place, the proofs of the theorems involve a rather elaborate induction on dimension of supports.

3.26. **Mixed Hodge modules.** Saito has given an extension of the previous theory by defining the notion of mixed Hodge module. I will start with recalling the older definition of a *variation of mixed Hodge structure*:

(1) A locally constant sheaf L of $\mathbb{Q}$ vector spaces with finite dimensional stalks.
(2) An ascending filtration $W \subset L$ by locally constant subsheaves
(3) A vector bundle with an integrable connection (E, ∇) plus an isomorphism $DR(E) \cong L \otimes \mathbb{C}[\dim Y]$.
(4) A filtration $F^\bullet$ of E by subbundles satisfying Griffiths' transversality: $\nabla(F^p) \subseteq F^{p-1}$.
(5) $(Gr^W_m(L), O_X \otimes Gr^W_m(L), F^\bullet(O_X \otimes Gr^W_m(L)))$ is a variation of pure Hodge structure of weight m.

The data induces a mixed Hodge structure on each of the stalks L_y, and hence the name. Steenbrink and Zucker [SZ] showed that additional conditions are required to get a good theory. While these conditions are rather technical, they do hold in most natural examples.

A variation of mixed Hodge structure over a punctured disk D^* is *admissible* if

(1) The pure variations $Gr^W_m(L)$ are polarizable.
(2) There exists a limit Hodge filtration $\lim F^p_t$ compatible with the one on $Gr^W_m(L)$ constructed by Schmid.
(3) There exists a so called relative monodromy filtration U on $(E = L_t, W)$ with respect to the logarithm N of the unipotent part of monodromy. This means that $NU_k \subseteq U_{k-2}$ and U induces the monodromy filtration on $Gr^W_k(E)$ constructed earlier up to suitable a shift. (Note that relative monodromy filtrations need not exist a priori.)

For a general base X the above conditions are required to hold for every restriction to a punctured disk [K1]. Note that pure variations of Hodge structure are automatically admissible.

Let me now turn to the general case. I will define a pre-mixed Hodge module on X to consist of

(1) A perverse sheaf L defined over $\mathbb{Q}$, together filtration W of L by perverse subsheaves.
(2) A regular holonomic D_X-module M with a filtration WM which corresponds to $(L \otimes \mathbb{C}, W \otimes \mathbb{C})$ under Riemann-Hilbert.
(3) A good filtration F on M.

These objects form a category, and Saito defines the subcategory of mixed Hodge modules $MHM(X)$ by a rather delicate induction. The key points are that for (M, F, L, W) to be in $MHM(X)$, we require

- the associated graded objects $Gr_k^W(M, F, L)$ yield polarizable Hodge modules of weight k,
- for any (germ of a) function f on X, the relative monodromy filtration U (resp. U') for $\psi_f(M, F, L)$ (resp. $\phi_{f,1}(M, F, L)$) with respect to W exists.
- The pre-mixed Hodge modules $(\psi_f(M, F, L), U)$ and $(\phi_{f,1}(M, F, L), U')$ are in fact mixed Hodge modules on $f^{-1}(0)$

The main properties are summarized below:

Theorem 3.27 (Saito)**.**
(1) *$MHM(X)$ is abelian, and it contains each $MH(X, n)^{pol}$ as a full abelian subcategory.*
(2) *$MHM(point)$ is the category of polarizable mixed Hodge structures.*
(3) *If $U \subseteq X$, then any admissible variation of mixed Hodge structure extends to an object in $MHM(X)$.*

Finally, we have:

Theorem 3.28 (Saito)**.** *There is a realization functor*

$$real : D^b MHM(X) \to D^b_{constr}(X, \mathbb{Q})$$

and refined direct image and inverse image operations

$$f_* : D^b MHM(X) \to D^b MHM(Y)$$
$$f^* : D^b MHM(Y) \to D^b MHM(X)$$

for each morphism $f : X \to Y$, such that

$$real(f_*\mathcal{M}) = \mathbb{R}f_* real(\mathcal{M})$$
$$real(f_*\mathcal{N}) = \mathbb{L}f_* real(\mathcal{M})$$

Similar statements hold for various other standard operations such as tensor products.

Putting this together with previous statements yields

Corollary 3.29. *The cohomology of a smooth variety U with coefficients in an admissible variation of mixed Hodge structure L carries a canonical mixed Hodge structure. The cup product*

$$H^i(U, L) \otimes H^j(U, L') \to H^{i+j}(U, L \otimes L')$$

is compatible with these structures.

When the base is a curve, this was first proved by Steenbrink and Zucker [SZ].

3.30. **Explicit construction.** I want to give a bit more detail on the construction of the mixed Hodge structure in corollary 3.29. Let U be a smooth n dimensional variety. We can choose a smooth compactification $j : U \to X$ such that $D = X - U$ is a divisor with normal crossings. Fix an admissible variation of mixed Hodge structure (L, W, E, F, ∇) on U. Extend (E, ∇) to a vector bundle E^I with logarithmic connection with the eigenvalues of its residues of the extension in I as in section 3.7 Let $M = \cup \bar{E}^I \subset j_*E$. This is a D_X-module which corresponds to the perverse sheaf $\mathbb{R}j_*L[n] \otimes \mathbb{C}$. Filter this by

$$F_pM = \sum_i F_iD_XF_{p-i}\bar{E}^{[-1,0)}$$

The rest of the story is somewhat more complicated, so I will first state the outcome:

Theorem 3.31. *There exists compatible filtrations $\tilde{W}$ on $\mathbb{R}j_*L[n]$ and M extending W over U such that $(\mathbb{R}j_*L[n], \tilde{W}, M, F)$ becomes a mixed Hodge module.*

This second filtration $\tilde{W}$ is not easy to describe. So I'll give an indication in the special case, where the original variation of mixed Hodge is pure and $D = f^{-1}(0)$ for some morphism $f : X \to \mathbb{P}^1$. In this case, $\mathbb{R}j_*L[n]$ fits into an exact sequence of perverse sheaves [E]

$$0 \to j_{!*}L[n] \to \mathbb{R}j_*L[n] \to K \to 0$$

where

$$K = coker(N : {}^p\psi_f L[n] \to {}^p\psi_f L[n])$$

To construct $\tilde{W}$, we can take $j_{!*}L[n]$ as the first step. The rest is obtained by pulling back (the image of) the monodromy filtration from K. In general, D is always given locally as $f^{-1}(0)$ for some f. Thus this description can be extended to the general case, once it is shown that the locally defined filtrations must patch. Note that it is always true that the associated graded $Gr_i^{\tilde{W}}$ decomposes into a direct sum of intersection cohomology complexes of pure polarized VHS's. This provides the crucial link to the earlier work of Cattani-Kaplan-Schmid and Kashiwara-Kawai (theorem 3.4).

Finally, let me sketch where the mixed Hodge structure comes from. We have an isomorphism

$$H^i(U, L \otimes \mathbb{C}) \cong \mathbb{H}^i(\Omega_X^\bullet(\log D) \otimes \bar{E}^{[0,1)})$$

and the filtrations defining the Hodge structure can be displayed rather explicitly from this. The filtration

$$F^p\Omega_X^\bullet(\log D) \otimes \bar{E}^{[0,1)} = \bigoplus_i \Omega_X^i \otimes F^{p-i}\bar{E}^{[0,1)}$$

induces the Hodge filtration on cohomology. We define a filtration $W_\bullet$ on this complex, by intersecting $(\Omega_X^\bullet \otimes \tilde{W}_{\bullet+n}M)[n]$ under the inclusion

$$\Omega_X^\bullet(\log D) \otimes \bar{E}^{[0,1)} \subset \Omega_X^\bullet \otimes M[n]$$

Then this filtration induces

$$W_{i+k}H^i(U, L \otimes \mathbb{C}) = \mathbb{H}^i(X, W_k\Omega_X^\bullet(\log D) \otimes \bar{E}^{[0,1)}).$$

Since we have a corresponding filtration on $\mathbb{R}j_*L$, it follows that W is defined over $\mathbb{Q}$. In more technical terms, this data constitutes a cohomological mixed Hodge complex.

3.32. **Real Hodge modules.** Saito has also developed a theory of (mixed) Hodge modules over the reals, where the underlying perverse sheaf is defined over $\mathbb{R}$. Some modifications in the basic set up are necessary (e.g. Kashiwara-Malgrange filtrations are now indexed by $\mathbb{R}$), but the basic theory goes through pretty much as before. In particular, the cohomology of real admissible variations of mixed Hodge structures carry real mixed Hodge structures.

4. Comparison of Hodge structures

The cohomological mixed Hodge complex described in the previous section reduces to the one given by Deligne [De3] for constant coefficients. Consequently:

Proposition 4.1. *Let X be smooth variety, then Saito's mixed Hodge structure (cor. 3.29) on $H^*(X,\mathbb{Q})$ agrees with Deligne's.*

From now on let $f : X \to Y$ be a smooth projective morphism of smooth quasiprojective varieties. We recall two results

Theorem 4.2 (Deligne [De1])**.** *The Leray spectral sequence*

$$E_2^{pq} = H^p(Y, R^q f_*\mathbb{Q}) \Rightarrow H^{p+q}(X, Q)$$

degenerates. In particular $E_2^{pq} \cong Gr_L^p H^{p+q}(X)$ for the associated "Leray filtration" L.

Theorem 4.3 ([A])**.** *There exists varieties Y_p and morphisms $Y_p \to Y$ such that*

$$L^p H^i(X,\mathbb{Q}) = ker[H^i(X,\mathbb{Q}) \to H^i(X_p,\mathbb{Q})]$$

where $X_p = f^{-1}Y_p$.

It follows that each L^p is a filtration by sub mixed Hodge structures. When combined with the earlier isomorphism, we get a mixed Hodge structure on $H^p(Y, R^q f_*\mathbb{Q})$ which I will call the naive mixed Hodge structure. On the other hand, $R^q f_*\mathbb{Q}$ carries a pure hence admissible variation of Hodge structure, so we can apply Saito's result 3.29 to get another Hodge structure.

Proposition 4.4. *The naive mixed Hodge structure coincides with Saito's.*

Proof. Deligne actually proved a stronger version of the above theorem which implies that

$$\mathbb{R}f_*\mathbb{Q} \cong \bigoplus R^i f_*\mathbb{Q}[-i] \tag{2}$$

(non canonically) in $D^b(X,\mathbb{Q})$. The theorem is quite general and thanks to 3.21, it even applies if regard these as objects in $D^bMHM(Y)$. Note that the Leray filtration is induced by the truncation filtration

$$L^p H^i(X,\mathbb{Q}) = image[H^i(Y, \tau_{\le i-p}\mathbb{R}f_*\mathbb{Q} \to H^i(Y, \mathbb{R}f_*\mathbb{Q})]$$

Under (2),

$$\tau_{\le p}\mathbb{R}f_*\mathbb{Q} \cong \bigoplus_{i\le p} R^i f_*\mathbb{Q}[-i]$$

Therefore we have a (non canonical!) isomorphism of mixed Hodge structures

$$H^i(X,\mathbb{Q}) \cong \bigoplus_{p+q=i} H^p(Y, R^p f_*\mathbb{Q})$$

where the right side is equipped with Saito's Hodge structure. Under this isomorphism, L^p maps to

$$H^{i-p}(R^p f_*\mathbb{Q}) \oplus H^{i-p+1}(R^{p-1} f_*\mathbb{Q}) \oplus \dots$$

The proposition now follows. ∎

This result goes back to Zucker [Z] when Y is a curve.

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