THE 3RD SYMPOSIUM ON ANALYSIS AND PDES

PURDUE UNIVERSITY, MAY 27–30, 2007

ABSTRACTS

Principal Lecturers

Gerhard Huisken, Max Planck Institute, Potsdam, Germany

Isoperimetric Inequalities Related to Mass and Energy in General Relativity

Asymptotically flat 3-manifolds of non-negative scalar curvature arise as 3-dimensional spacelike (maximal) slices of Lorentzian 4-manifolds modelling isolated gravitating systems such as stars, binaries, galaxies and black holes. Traditionally the total mass of an isolated system is defined by a flux integral near infinity involving first derivatives of the asymptotically flat metric. The course explains how this concept of mass can be replaced both locally and globally by by a more geometric concept describing the mass as a defect in the classical isoperimetric inequality. Using geometric evolution equations such as mean curvature flow and inverse mean curvature flow we show that the isoperimetric concept is compatible with the more classical concepts of ADM-mass and Hawking-mass and derive positive lower bounds related to the positive mass theorem and the Penrose inequality.

Lectures:

1. Concepts for mass and energy of isolated systems in General Relativity.
2. Geometric evolution equations and monotonicity formulae.
3. Isoperimetric inequalities via geometric evolution equations and the isoperimetric concept of mass.
4. Positive mass theorems for the isoperimetric mass and a concept of quasilocal isoperimetric mass.

Juan Luis Vázquez, Universidad Autónoma de Madrid, Spain

Four Lectures on the Theory of Fast Diffusion

The minicourse is aimed at presenting some of the recent progress in the mathematical theory of nonlinear diffusion processes by focusing on the model called the fast diffusion equation.

Lectures:

1. Introduction to Linear and Nonlinear Diffusion. Main models in problems of viscous fluids, phase change, water infiltration, population dynamics, and plasma physics.

   The Fast Diffusion Equation and the Porous Medium Equation. Main features. Slow and fast propagation.

2. Existence of different types of solutions. Well-posedness. Existence in optimal classes of data. Cases of non-uniqueness and non-existence. The role of critical exponents.

   Regularity of the solutions: smoothing effects and continuity.
3. Asymptotic behaviour of the solutions for large time. Extinction. Special selfsimilar solutions.
4. The geometrical models. The evolution flows: Yamabe ($n = 3$) and Ricci ($n = 2$).

   New lines of research: p-Laplacians and fast diffusion.

References:

A large part of the material is taken from the following monograph:

- J. L. Vázquez, *Smoothing and decay estimates for nonlinear diffusion equations*, vol. **33** of Oxford Lecture Notes in Maths. and its Applications, Oxford Univ. Press, 2006.

Invited Speakers

Patricia Bauman, Purdue University, West Lafayette, IN

A Variational Model for Bent-Core Liquid Crystals

We present a nonlinear variational model for bent-core (banana-shaped) liquid crystals which contains energy terms arising from polarization, smectic and chiral effects, elasticity, and surface tension. Bent-core liquid crystals are made of molecules that are bent rather than straight (rod-like) as in standard liquid crystals. Their structure is described by energy minimizers.

We analyze stable solutions in a physically realistic regime. The mathematical analysis involves a free boundary problem for the energy and the existence of a Gamma-limit.

Panagiota Daskalopoulos, Columbia University, New York, NY

On the Optimal Regularity of Solutions to Two Dimensional Monge-Ampére Equation

We will discuss the optimal regularity of solutions to degenerate Monge-Ampére equations

$$\det D^2 u = f(x, u, \nabla u), \qquad \text{on } \Omega \subset R^2$$

in the case where $f \geq 0$ and vanishes at points or interfaces.

Inwon Kim, University of California, Los Angeles, CA

Regularity and Monotonicity of Hele-Shaw Flow

In this talk we will discuss regularization of one-phase Hele-Shaw flow, starting from initially Lipschitz surface. We show that if the Lipschitz constant of the initial free boundary is small, then for small positive time the solution of (HS) is smooth and solves (HS) in the classical sense. Global and local version of the result will be presented with estimates on the free boundary speed in terms of the initial data. Existence, uniqueness and monotonicity properties of global Hele-Shaw flow will follow. This is joint work with Sunhi Choi and David Jerison.

Juan J. Manfredi, University of Pittsburgh, Pittsburgh, PA

Sharp Friedrichs-Knapp-Stein Inequalities and Applications

The classical Friedrichs identity states that for $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |D^2 u|^2 \, dx = \int_{\mathbb{R}^n} |\Delta u|^2 \, dx.$$

From this inequality we immediately get $W^{2,2}$-estimates for solutions of $\Delta u = f$ and also for solutions of measurable perturbations of the form $\sum_{ij} a_{ij}(x) u_{ij}(x) = f(x)$, when the matrix $A = (a_{ij})$ is closed to the identity in sense made precise by Cordes. In this talk we explore extensions of the Friedrichs identity in the form of sharp inequalities

$$\int_X |\mathfrak{X}^2 u|^2 \, dx \leq C_1 \int_X |\Delta_{\mathfrak{X}} u|^2 \, dx + C_2 \int_X |\mathfrak{X}|^2 \, dx$$

where X is a Riemannian manifold, a nilpotent Lie group, and a CR manifold.

This is joint work with Andràs Domokos (PAMS 133, 2005) and Sagun Chanillo (2007 preprint.)

Lei Ni, University of California, San Diego, CA

Connecting the Entropy Formulae

We shall explain an interpolation which connects Hamilton's entropy formula with Perelman's.

Dejan Slepcev, Carnegie Mellon University, Pittsburgh, PA

Interfacial Dynamics in a Model of Biological Aggregation

I will discuss some features of a continuum model of biological aggregation introduced by Topaz, Bertozzi and Lewis. Individuals, described by the model, experience long range attraction to other individuals of their species, but try to avoid overcrowding. In the continuum model for the population density the attraction is described via a nonlocal operator, while the repulsion is modeled by a differential operator.

I will describe why such behavior leads to formation of interfaces between a near-constant-density aggregate state and the unpopulated space. The dynamics of the interfaces, and why surface-tension-like effects occur at the interfaces will be discussed.

On long time scales the interfacial motion leads to coarsening of length scales in the evolution of a typical configuration. The rate of coarsening can be studied using the Kohn-Otto framework. I will describe a geometric viewpoint that unites the coarsening results in a variety of interfacial models.

The talk is partially based on joint work with Andrea Bertozzi.

Lihe Wang, University of Iowa, Iowa City, IA

Estimates for a Class of Degenerate Equations

Brian White, Stanford University, Stanford, CA

Singularities in Mean Curvature Flow

An overview of what we know about partial regularity and singularity structure in mean curvature flow, and some open problems.

Yu Yuan, University of Washington, Seattle, WA

Hessian Estimates for Special Lagrangian Equations

We survey some recent results on Hessian estimates for special Lagrangian equations, including the σ_2 equation in dimension three. The gradient graph of the solutions are minimal Lagrangian surfaces. An Hessian estimate for the σ_2 (Monge-Ampere) equation in dimension two was obtained by Heinz in the 1950's, and irregular solutions to the σ_3 (Monge-Ampere) equation in dimension three were constructed by Pogorelov in the 1970's. This is joint work with Micah Warren.

Contributed Speakers

Anahit Galstyan, University of Texas - Pan American, Edinburg, TX

On the Oscillations of the Solution Curve for a Class of Semilinear Equations

We use the bifurcation theory to investigate a number of positive solutions of the semilinear Dirichlet boundary value problem on a ball in Rn for the second order elliptic equation with periodic nonlinearity containing a positive parameter. We show that if the dimension n of the variables is less than or equal 5, the problem has infinitely many positive solutions when the parameter coincides with the principal eigenvalue of the Laplace operator. At any other value of the parameter or if the dimension n exceeds 5, the number of positive solutions is at most finite. Our approach appeals to the well known results of B. Gidas, W.-M. Ni, L. Nirenberg, the bifurcation theorems of M.G.Crandall, and P.H.Rabinowitz and the stationary phase method.

The presentation is based on the joint paper with Philip Korman, University of Cincinnati and Yi Li, University of Iowa.

Luan Thach Hoang, University of Minnesota, Minneapolis, MN

Regularity of the Stokes Operator in Thin Domains

The problem arises in the study of the Navier-Stokes equations with Navier friction boundary conditions in a 3D thin domain Ω_ε which has thickness of order $O(\varepsilon)$ as $\varepsilon \to 0$, and non-flat top and bottom boundaries. Our aim is to obtain the estimate $\|u\|_{H^2(\Omega_\varepsilon)} \le C_\varepsilon \|Au\|_{L^2(\Omega_\varepsilon)}$, with explicit dependence of C_ε on ε, where u belongs to the domain D_A of the Stokes operator A which is related to the mentioned boundary conditions. Because of the boundary's non-trivial geometry and the involved boundary conditions, the constant C_ε is not known to have a uniform bound when $\varepsilon \to 0$. Our result is the following estimate

$$\|u\|_{H^2(\Omega_\varepsilon)} \le c\|Au\|_{L^2(\Omega_\varepsilon)} + c_\varepsilon \|u\|_{L^2(\Omega_\varepsilon)}, \quad u \in D_A,$$

where c is independent of ε, the positive number c_ε depends explicitly on ε and the friction coefficients, and may blow up as $\varepsilon \to 0$. (This estimate is used in the theory of the global strong solutions to the Navier-Stokes equations considered above.)

Qinian Jin, University of Texas at Austin, Austin, TX

Some Conformally Invariant Fully Nonlinear Elliptic Equations on Manifolds with Boundary

We consider, on compact manifolds with boundary, some conformally invariant fully nonlinear elliptic equations related to the schouten tensor together with some boundary mean curvature conditions. This is a fully nonlinear version of the Yamabe problem on manifolds with boundary. We will present some local gradient and hessian estimates under suitable conditions. Moreover, we will address some existence results.

Xuan Hien Nguyen, University of Cincinnati, Cincinnati, OH

Construction of Complete Embedded Self-Similar Surfaces Under Mean Curvature Flow

We carry out the first main steps towards the construction of new examples of complete embedded self similar surfaces under mean curvature flow. The general strategy is inspired by the work of N. Kapouleas: a surface that is close to being self similar is obtained by taking two known examples of self similar surfaces, a cylinder and plane, and desingularizing the circle at the intersection using an appropriately modified singly periodic Scherk surface. This approximation will then be perturbed to obtain a self-similar surface. We will present some promising results towards the implementation of the above strategy.

Giuseppe Tinaglia, University of Notre Dame, Notre Dame, IN

Multi-Valued Graphs in Embedded Constant Mean Curvature Disks

We prove that an embedded and simply connected constant mean curvature surface with Gaussian curvature large at a point contains a multi-valued graph around that point on the scale of the norm squared of the second fundamental form. This generalizes Colding-Minicozzi result for minimal surfaces.

Tung To, Columbia University, New York, NY

A Free-Boundary Problem for the Evolution p-Laplacian Equation with Heat Combustion Boundary Condition

We consider the following free-boundary problem: given a non-negative function u_0 defined in $\mathbb{R}^n$, whose positivity set is a bounded domain Ω_0, find a domain $\Omega \subset \mathbb{R}^n \times [0,T]$ for some positive T and a function $u(x,t)$ such that

$$\begin{cases} u_t = \Delta_p u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 \text{ and } |Du| = 1 & \text{on } \partial\Omega \cap \{0 < t < T\} \\ u(\cdot, 0) = u_0 & \text{in } \Omega_0 \end{cases} \tag{1}$$

The operator

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2}\, Du\right)$$

is known as the p-Laplacian.

This problem in the case $p = 2$ has been studied by Caffarelli and Vazquez. In this work, I will prove the following result in the case $p > 2$:

THEOREM. *Let Ω_0 be a bounded convex domain whose boundary is $C^{1+\alpha}$. The function $u_0(x)$ is concave and in $C^{1+\alpha}(\Omega_0)$. Then there is a solution (u, Ω) of the problem* (1) *which exists up to a vanishing time T.*

Moreover, the vanishing time T is finite, the free-boundary $\partial\Omega_t$ is in C^∞ for positive t and this solution is unique.

REFERENCES:

- L.A.Caffarelli and J.L.Vazquez, *A free boundary problem for the heat equation arising in flame propagation*, Trans. Amer. Math. Soc.**347** (1995), 411-441.