MA366 FINAL ADDITIONAL PRACTICE PROBLEM SOLUTIONS

1. Rewrite the second order equation

$$2u'' + 3u' + ku = \cos 2t$$

as a system of first order equations.

Solution. Let $x = u$, $y = u'$. Then

$$\begin{aligned} x' &= \boxed{y} \\ y' = u'' &= \frac{1}{2}(-3u' - ku + \cos 2t) \\ &= \boxed{\frac{1}{2}(-3y - kx + \cos 2t).} \end{aligned}$$

□

2. The solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

is?

Solution. Denote the matrix by A.

- Eigenvalues of A.

$$\begin{aligned} &\det(A - rI) = r^2 - 2r - 3 = 0 \\ &(r-3)(r+1) = 0 \\ &r_1 = 3, \quad r_2 = -1 \end{aligned}$$

- Eigenvectors.

$$\begin{aligned} r_1 = 3: \quad &(A - 3I)\xi = 0 \\ &-2\xi_1 + \xi_2 = 0 \\ &\xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} r_2 = -1: \quad &(A + I)\xi = 0 \\ &2\xi_1 + \xi_2 = 0 \\ &\xi^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned}$$

- General solution.

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

- Solution of IVP.

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$c_1 + c_2 = 3; \quad 2c_1 - 2c_2 = 2$$

$$c_1 = 2; \quad c_2 = 1.$$

Hence

$$\boxed{\mathbf{x} = 2\begin{pmatrix}1\\2\end{pmatrix}e^{3t} + \begin{pmatrix}1\\-2\end{pmatrix}e^{-t}.}$$

□

3. Solve

$$\mathbf{x}' = \begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix}-1\\2\end{pmatrix}.$$

Solution. Denote the matrix by A.

- Eigenvalues of A.

$$\begin{aligned}\det(A - rI) &= r^2 - 2r + 2 = 0\\ r = 1 \pm i &\quad \textit{complex eigenvalues}\end{aligned}$$

- Complex eigenvector.

$$\begin{aligned} r = 1 + i: \quad (A - (1+i)I)\xi &= 0\\ -i\xi_1 + \xi_2 &= 0\\ \xi &= \begin{pmatrix}1\\i\end{pmatrix}\end{aligned}$$

- Complex-valued solution.

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix}1\\i\end{pmatrix}e^{(1+i)t}\\ &= \left[\begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix}\right]e^t(\cos t + i\sin t)\\ &= e^t\left[\begin{pmatrix}1\\0\end{pmatrix}\cos t - \begin{pmatrix}0\\1\end{pmatrix}\sin t\right]\\ &+ ie^t\left[\begin{pmatrix}0\\1\end{pmatrix}\cos t + \begin{pmatrix}1\\0\end{pmatrix}\sin t\right].\end{aligned}$$

- Real-valued solutions.

$$\begin{aligned}\mathbf{x}^{(1)} &= e^t\begin{pmatrix}\cos t\\-\sin t\end{pmatrix}\\ \mathbf{x}^{(2)} &= e^t\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}\end{aligned}$$

- General solution.

$$\mathbf{x} = e^t\left[c_1\begin{pmatrix}\cos t\\-\sin t\end{pmatrix} + c_2\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}\right]$$

- Solution of IVP

$$\mathbf{x}(0) = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$$

$$c_1 = -1; \quad c_2 = 2$$

$$\boxed{\mathbf{x} = e^t\left[-\begin{pmatrix}\cos t\\-\sin t\end{pmatrix} + 2\begin{pmatrix}\sin t\\\cos t\end{pmatrix}\right].}$$

□

4. Solve the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix}1\\1\end{pmatrix},$$

where $A = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$.

Solution.

- Eigenvalues of A

$$\det(A - rI) = r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1 \quad \textit{repeated eigenvalue}$$

- Eigenvector(s)

$$r = 1: \quad (A - I)\xi = 0$$

$$\xi_2 = 0$$

$$\xi = \begin{pmatrix}1\\0\end{pmatrix}.$$

- Solutions

$$\mathbf{x}^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}e^t$$

$$\mathbf{x}^{(2)} = \begin{pmatrix}1\\0\end{pmatrix}te^t + \eta e^t,$$

where η solves

$$(A - I)\eta = \xi$$

$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}\eta_1\\\eta_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\eta_2 = 1$$

and we are free to choose $\eta_1 = 0$. Thus, $\eta = \begin{pmatrix}0\\1\end{pmatrix}$ and

$$\mathbf{x}^{(2)} = \begin{pmatrix}1\\0\end{pmatrix}te^t + \begin{pmatrix}0\\1\end{pmatrix}e^t.$$

- General solution

$$\mathbf{x} = c_1\begin{pmatrix}1\\0\end{pmatrix}e^t + c_2\left[\begin{pmatrix}1\\0\end{pmatrix}te^t + \begin{pmatrix}0\\1\end{pmatrix}e^t\right].$$

• Solution of IVP

$$\begin{aligned}
\mathbf{x}(0) &= c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}\\
c_1 &= 1, \quad c_2 = 1\\
\mathbf{x} &= \begin{pmatrix}1\\0\end{pmatrix}e^t + \left[\begin{pmatrix}1\\0\end{pmatrix}te^t + \begin{pmatrix}0\\1\end{pmatrix}e^t\right]\\
&= \boxed{\begin{pmatrix}1\\1\end{pmatrix}e^t + \begin{pmatrix}1\\0\end{pmatrix}te^t.}
\end{aligned}$$

□

5. Find a particular solution of

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}' = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix}.$$

Solution. Using the method of undetermined coefficients. Denote the matrix by A. Eigenvalues of A are

$$\begin{aligned}
\det(A - rI) &= r^2 - 1 = 0\\
r &= \pm 1.
\end{aligned}$$

Since $r = 0$ is not an eigenvalue, the particular solution has the form

$$\mathbf{x}_p = \mathbf{a} = \begin{pmatrix}a_1\\a_2\end{pmatrix}.$$

We have

$$\begin{aligned}
\mathbf{a}' = 0 &= A\mathbf{a} - \begin{pmatrix}2\\3\end{pmatrix}\\
\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} &= \begin{pmatrix}2\\3\end{pmatrix}\\
a_1 = 3; \quad a_2 &= 2.
\end{aligned}$$

Thus,

$$\boxed{\mathbf{x}_p = \begin{pmatrix}3\\2\end{pmatrix}.}$$

□

6. Find the general solution of

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}' = \begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} - \begin{pmatrix}6e^{-t}\\1\end{pmatrix}.$$

Solution. Denote the matrix by A.

• Eigenvalues of A.

$$\begin{aligned}
\det(A - rI) &= r^2 - 3r + 2 = 0\\
(r-2)(r-1) &= 0\\
r_1 = 1, \quad r_2 &= 2
\end{aligned}$$

- Eigenvectors.

$$\begin{aligned} r_1 = 1: \quad & (A - I)\xi = 0 \\ & \xi_1 = 0 \\ & \xi^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} r_2 = 2: \quad & (A - 2I)\xi = 0 \\ & \xi_1 - \xi_2 = 0 \\ & \xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

- General solution of homogeneous equation

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

- Particular solution.

Using the method of undetermined coefficients. Rewrite the free term as

$$\begin{pmatrix} 6e^{-t} \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $r = -1$ and $r = 0$ are not eigenvalues, the particular solution has the form

$$\mathbf{x}_p = \mathbf{a}e^{-t} + \mathbf{b}.$$

We have

$$\begin{aligned} & (\mathbf{a}e^{-t} + \mathbf{b})' - A(\mathbf{a}e^{-t} + \mathbf{b}) = \\ & (-\mathbf{a} - A\mathbf{a})e^{-t} - A\mathbf{b}, \end{aligned}$$

hence the following equalities should hold

$$\begin{aligned} (A + I)\mathbf{a} &= \begin{pmatrix} 6 \\ 0 \end{pmatrix} \\ A\mathbf{b} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Equivalently,

$$\begin{aligned} 3a_1 = 6; \quad & a_1 + 2a_2 = 0 \\ 2b_1 = 0; \quad & b_1 + b_2 = 1, \end{aligned}$$

which gives

$$\begin{aligned} a_1 = 2; \quad & a_2 = -1 \\ b_1 = 0; \quad & b_2 = 1. \end{aligned}$$

Thus, the general solution is

$$\boxed{c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$\square$