# MA366 MIDTERM EXAM 1 PRACTICE PROBLEM SOLUTIONS

**1.** If $y' + (1 + \frac{1}{t})y = \frac{1}{t}$ and $y(1) = 0$, then $y(\ln 2) = ?$

*Solution.* Using the method of integrating factors:

$$\mu(t) = \exp \int \left(1 + \frac{1}{t}\right) dt = \exp(t + \ln|t| + c) = Cte^t.$$

We take $C = 1$. Multiplying the equation by $\mu(t)$, we obtain

$$(te^t y)' = e^t$$
$$te^t y = e^t + C$$
$$y = \frac{1}{t} + \frac{C}{te^t}.$$

Plugging $t = 1$ and verifying that $y(1) = 0$, we obtain

$$y(1) = 1 + \frac{C}{e} \quad \Rightarrow \quad C = -e$$
$$y = \frac{1}{t} - \frac{e}{te^t}.$$

Thus,

$$y(\ln 2) = \frac{1}{\ln 2} - \frac{e}{2\ln 2} = \boxed{\frac{1}{\ln 2}\left(1 - \frac{e}{2}\right)}$$

□

**2.** What is the largest open interval for which a unique solution of the initial value problem

$$ty' + \frac{1}{t+1}y = \frac{t-2}{t-3}, \quad y(1) = 0$$

is guaranteed?

*Solution.* This is a linear equation, and dividing by $t$ we will bring it to the form

$$y' + \frac{1}{t(t+1)}y = \frac{t-2}{t(t-3)},$$

for which we can apply the Existence and Uniqueness Theorem. Namely, the interval in questions is the largest interval containing $t_0 = 1$, where functions $p(t) = \frac{1}{t(t+1)}$ and $g(t) = \frac{t-2}{t(t-3)}$ have no discontinuities. Those functions are discontinuous at $t = -1, 0, 3$, hence the largest interval of uniqueness and existence for our problem is $\boxed{0 < t < 3}$. □

**3.** Consider the autonomous differential equation

$$\frac{dy}{dt} = -\frac{1}{10}(y-1)(y-4)^2.$$

Classify the stability of each equilibrium solution.

*Solution.* Let $f(y) = -(1/10)(y-1)(y-4)^2$. The equilibrium solutions are $y = 1$ and $y = 4$, which are the roots of $f(y) = 0$. To determine the stability of the equilibriums we analyze the sign of $y'$.

$$\begin{aligned} 4 < y < \infty: &\quad y' = f(y) < 0 \quad y \downarrow \\ 1 < y < 4: &\quad y' = f(y) < 0, \quad y \downarrow \\ -\infty < y < 1: &\quad y' = f(y) > 0, \quad y \uparrow \end{aligned}$$

This tells us that $\boxed{y = 1 \text{ is stable}}$ and $\boxed{y = 4 \text{ is unstable}}$ (actually semistable). □

**4.** Determine whether

$$x + 2y + (2x + y)\frac{dy}{dx} = 0$$

is separable, linear and/or exact.

*Solution.* Let

$$M(x, y) = x + 2y$$
$$N(x, y) = 2x + y$$

The equation is

– Not separable, since not of the form $M(x) + N(y)\frac{dy}{dx} = 0$;

– Not linear, since contains the term $y\frac{dy}{dx}$.

– $\boxed{\text{Exact}}$, since $M_y = 2 = N_x$. □

**5.** An explicit solution of $y' = y^2 - 1$ is?

*Solution.* Using the method of separation of variables.

$$\frac{dy}{dt} = y^2 - 1$$
$$\frac{dy}{y^2 - 1} = dt$$
$$\int \frac{dy}{y^2 - 1} = \int dt$$

Now, using the partial fraction decomposition

$$\frac{1}{y^2 - 1} = \frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right),$$

we obtain

$$\frac{1}{2}(\ln|y-1| - \ln|y+1|) = t + c$$
$$\frac{y-1}{y+1} = Ce^{2t}, \quad C = \pm e^{2c}$$
$$\boxed{y = \frac{1 + Ce^{2t}}{1 - Ce^{2t}}.}$$

□

**6.** If $y' = y^3$ and $y(0) = 1$, then $y(-1) = ?$

*Solution.* Separation of variables.

$$\int \frac{dy}{y^3} = \int dt$$
$$-\frac{1}{2y^2} = t + C$$
$$y = \pm \frac{1}{\sqrt{-2(t + C)}}.$$

From the initial condition

$$y(0) = \pm \frac{1}{\sqrt{-2C}} = 1.$$

Hence $C = -1/2$ and we have to take + sign in front of the square root.

$$y = \frac{1}{\sqrt{1 - 2t}}.$$

Thus,

$$\boxed{y(-1) = \frac{1}{\sqrt{3}}}$$

□

**7.** If

$$\left(\frac{2}{x} + \frac{y^2}{x^3}\right) dx - \frac{y}{x^2} dy = 0$$

and $y(1) = 2$, then $y(e^3) = ?$

*Solution.* This is an exact equation. Indeed, let

$$M(x, y) = \frac{2}{x} + \frac{y^2}{x^3}$$
$$N(x, y) = -\frac{y}{x^2}.$$

Then

$$M_y = \frac{2y}{x^3} = N_x$$

so the equation is exact. Let us now find $\psi$ such that

$$\psi_x = M = \frac{2}{x} + \frac{y^2}{x^3}$$
$$\psi_y = N = -\frac{y}{x^2}$$

Integrating the first equation gives

$$\psi = 2\ln|x| - \frac{y^2}{2x^2} + h(y).$$

To find $h(y)$, we use the second equation for $\psi$:

$$\psi_y = -\frac{y}{x^2} + h'(y) = -\frac{y}{x^2}.$$

This implies $h'(y) = 0$, hence we can take $h(y) = 0$. So the implicit solution is given by

$$\psi(x, y) = 2\ln|x| - \frac{y^2}{2x^2} = C.$$

Solving for $y$, we obtain

$$y = \pm 2x\sqrt{\ln|x| + C}.$$

From the initial condition, we obtain

$$y(1) = \pm 2\sqrt{C} = 2$$
$$C = 1$$
$$y = 2x\sqrt{\ln|x| + 1}.$$

Thus,

$$y(e^3) = 2e^3\sqrt{3 + 1} = \boxed{4e^3}.$$

□

**8.** An implicit solution of
$$y^2 + 1 + (2xy + 1)\frac{dy}{dx} = 0$$
is?

*Solution.* This is an exact equation. Indeed, let
$$M(x, y) = y^2 + 1$$
$$N(x, y) = 2xy + 1.$$
Then
$$M_y = 2y = N_x.$$
Next we find a function $\psi(x, y)$ such that
$$\psi_x = M = y^2 + 1$$
$$\psi_y = N = 2xy + 1.$$
Integrating the first equation, we have
$$\psi = \int (y^2 + 1)dx = x(y^2 + 1) + h(y),$$
where $h(y)$ to be found. Differentiating w.r.t. $y$, we obtain
$$\psi_y = 2xy + h'(y) = 2xy + 1$$
$$h'(y) = 1, \quad h(y) = y$$
$$\psi = xy^2 + x + y.$$
Thus, the solution is implicitly given by
$$\boxed{xy^2 + x + y = C.}$$
□

**9.** If $y'$ is proportional to $y$, $y(0) = 2$ and $y(1) = 8$, for what value of $t$ does $y(t) = 20$?

*Solution.* We have $y' = ky$, hence
$$y = Ce^{kt}.$$
We have
$$y(0) = C = 2 \quad \Rightarrow \quad C = 2$$
$$y(1) = 2e^k = 8 \quad \Rightarrow \quad k = \ln 4.$$
Hence
$$y = 2(e^{\ln 4t})$$
and
$$y = 20 \quad \Rightarrow \quad \boxed{t = \frac{\ln 10}{\ln 4}.}$$
□

**10.** Solve the initial value problem
$$y'' - 4y' + 4y = 0$$
$$y(0) = 1, \quad y'(0) = -1.$$

*Solution.* The characteristic equation is
$$r^2 - 4r + 4 = (r - 2)^2 = 0.$$
So, $r = 2$ is a root of multiplicity 2. Thus,
$$y_1 = e^{2t}, \quad y_2 = te^{2t}$$
form a fundamental set of solutions and the general solution is given by
$$y = e^{2t}(C_1 + C_2 t).$$
Now, it remains to find the constants $C_1$ and $C_2$ from the initial conditions. We first find
$$y' = e^{2t}(2C_1 + C_2 + 2C_2 t).$$
Then the initial conditions become
$$y(0) = C_1 = 1$$
$$y'(0) = 2C_1 + C_2 = -1.$$
This gives $C_1 = 1$, $C_2 = -3$ and
$$\boxed{y = e^{2t}(1 - 3t)}.$$
□

**11.** The general solution of
$$y'' + 4y' + 5y = 0$$
is?

*Solution.* The characteristic equation is
$$r^2 + 4r + 5 = 0$$
and the roots are
$$r = -2 \pm i.$$
The corresponding solutions are
$$y_1 = e^{-2t}\cos t, \quad y_2 = e^{-2t}\sin t,$$
and the general solution is given by
$$\boxed{y = e^{-2t}(C_1 \cos t + C_2 \sin t).}$$
□

**12.** A particular solution, $Y$, of
$$y'' - 4y' + 3y = 2t + e^t$$
is?

*Solution.* The characteristic equation is

$$r^2 - 4r + 3 = (r-3)(r-1) = 0$$

and the roots are $r = 1$ and $r = 3$. Using the method of undetermined coefficients, the form of $Y$ is

$$Y = ate^t + bt + c.$$

Then we have

$$\begin{aligned} Y' &= ate^t + ae^t + b \\ Y'' &= ate^t + 2ae^t \end{aligned}$$

$$Y'' - 4Y' + 3Y = -2ae^t + 3bt + 3c - 4b.$$

Hence

$$-2a = 1; \quad 3b = 2; \quad 3c - 4b = 0$$
$$a = -\frac{1}{2}; \quad b = \frac{2}{3}; \quad c = \frac{8}{9}.$$

So,

$$\boxed{Y = -\frac{1}{2}te^t + \frac{2}{3}t + \frac{8}{9}.}$$

□

**13.** If $y'' + 5y' + 6y = 24e^t$, $y(0) = 0$, $y'(0) = 0$, then $y(1) = ?$

*Solution.* Using the method of undetermined coefficients. The characteristic equation is

$$r^2 + 5r + 6 = (r+2)(r+3) = 0.$$

This gives that

$$y_1 = e^{-2t}, \quad y_2 = e^{-3t}$$

form a fundamental set of solutions of the homogeneous problem.

Next, since $r = 1$ is not a root of the characteristic equation, we look for a particular solution $Y$ of the form

$$Y = Ae^t.$$

We must have

$$Y'' + 5Y' + 6Y = (A + 5A + 6A)e^t = 24e^t,$$

which gives $A = 2$. Thus, $Y = 2e^t$ is a particular solution and

$$y = c_1e^{-2t} + c_2e^{-3t} + 2e^t$$

is the general one. Veryfying the initial conditions, we obtain

$$\begin{aligned} y(0) &= c_1 + c_2 + 2 = 0 \\ y'(0) &= -2c_1 - 3c_2 + 2 = 0, \end{aligned}$$

which gives $c_1 = -8$ and $c_2 = 6$. Hence

$$\boxed{y(1) = -8e^{-2} + 6e^{-3} + 2e.}$$

□

**14.** The differential equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0$$

has solutions $y_1(t) = t$ and $y_2(t) = t^2$. If

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 2; \quad y(1) = 0, \; y'(1) = 0$$

then $y(2) = ?$

*Solution.* Using the method of variation of parameters. A particular solution of the equation, which also satisfies the initial condition $y(1) = 0$, $y(1)' = 0$ is given by

$$\begin{aligned} y &= u_1(t)y_1 + u_2(t)y_2 \\ u_1(t) &= -\int_1^t \frac{y_2(\tau)g(\tau)}{W}d\tau \\ u_2(t) &= \int_1^t \frac{y_1(\tau)g(\tau)}{W}d\tau, \end{aligned}$$

where

$$W = y_1y_2' - y_2y_1' = t^2$$

is the Wronskian and $g(t) = 2$. We have

$$\begin{aligned} u_1(t) &= -\int_1^t 2d\tau = 2 - 2t \\ u_2(t) &= \int_1^t \frac{2}{\tau}\,d\tau = 2\ln t. \end{aligned}$$

Thus,

$$y = (2 - 2t)t + 2t^2 \ln t$$

and

$$y(2) = \boxed{8\ln 2 - 4.}$$

□

**15.** A tank initially contains 40 ounces of salt mixed in 100 gallons of water. A solution containing 4 oz of salt per gallon is ten pumped into the tank at the rate of 5 gal/min. The stirred mixture flows out of the tank at the same rate. How much salt is in the tank after 20 minutes?

*Solution.* Let $Q(t)$ be the amount (in oz) of salt in the tank at time $t$ (in min). Then

$$\begin{aligned}\frac{dQ}{dt} &= \text{Rate}_{\text{in}} - \text{Rate}_{\text{out}}\\ &= 4 \cdot 5 - \frac{Q}{100} \cdot 5.\end{aligned}$$

Thus we obtain an initial value problem

$$\frac{dQ}{dt} = 20 - \frac{Q}{20}; \quad Q(0) = 40.$$

Using the method of integrating factors

$$\begin{aligned}&\frac{dQ}{dt} + \frac{dQ}{20} = 20\\ &\frac{d}{dt}(e^{t/20}Q) = 20e^{t/20}\\ &e^{t/20}Q = 400e^{t/20} + C\\ &Q = 400 + Ce^{-t/20}.\end{aligned}$$

Since at $Q(0) = 40$, we obtain $C = -360$ and

$$Q = 400 - 360e^{-t/20}.$$

Thus $\boxed{Q(20) = 400 - 360e^{-1}.}$ □