MA366 Midterm Exam 1

Date: October 4, 2007 *Duration:* 75 min

Name: ____________________

PUID: ____________________

- Please fill in your name and your PUID in the space above.
- Remember, all answers must be justified and you must show all your work.

Problem	**Points**	**Score**
1	16	
2	16	
3	17	
4	18	
5	17	
6	16	
Total:	100	

1. A tank initially contains 100 gallons of fresh water. Then salt water containing 2 lbs of salt per gallon is pumped in at a rate of 4 gallons per minute, and the well-mixed mixture is allowed to leave at the same rate. How many pounds of salt there will be in the tank after 25 min? [16pt]

Solution: Let $Q(t)$ be the amount (in gallons) of salt in the tank at time t (in min). Then

$$\begin{aligned}\frac{dQ}{dt} &= \text{Rate}_{\text{in}} - \text{Rate}_{\text{out}} \\ &= 2 \cdot 4 - \frac{Q}{100} \cdot 4.\end{aligned}$$

Thus we obtain an initial value problem

$$\frac{dQ}{dt} = 8 - \frac{Q}{25}; \quad Q(0) = 0.$$

Using the method of integrating factors

$$\begin{aligned}&\frac{dQ}{dt} + \frac{dQ}{25} = 8 \\ &\frac{d}{dt}(e^{t/25}Q) = 8e^{t/25} \\ &e^{t/25}Q = 200e^{t/25} + C \\ &Q = 200 + Ce^{-t/25}.\end{aligned}$$

Since at $Q(0) = 0$, we obtain $C = -200$ and

$$Q = 200(1 - e^{-t/25}).$$

Thus $Q(25) = 200(1 - e^{-1})$.

2. Solve the initial value problem [16pt]

$$(t^2+1)\frac{dy}{dt} + 4ty = 1, \qquad y(0) = 5.$$

Solution: This is a first order linear equation, so we can use the method of integrating factors.

First, normalize the equation by dividing by the coefficient of the highest order term

$$\frac{dy}{dt} + \frac{4t}{t^2+1}y = \frac{1}{t^2+1}.$$

Now, the integration factor is given by

$$\mu(t) = \exp\left(\int \frac{4t}{t^2+1}\right) = \exp\left(2\int \frac{d(t^2+1)}{t^2+1}\right) = \exp\big(2\ln(t^2+1)+c\big) = C(t^2{+}1)^2.$$

We can take $C = 1$ in the above formula. Now, multiplying by μ, we obtain

$$((t^2+1)^2 y)' = t^2+1$$
$$(t^2+1)^2 y = \frac{1}{3}t^3 + t + C$$
$$y = \frac{\frac{1}{3}t^3 + t + C}{(t^2+1)^2},$$

where C to be found from the initial condition. We have

$$y(0) = C = 5$$

so the solution of the initial value problem is

$$y = \frac{\frac{1}{3}t^3 + t + 5}{(t^2+1)^2},$$

3. Show that the following equation is exact and then solve the initial value problem [17pt]

$$\left(\frac{y}{x} + 6x\right) + (\ln x - 2)\frac{dy}{dx} = 0, \quad y(1) = 3.$$

Write your answer in the form $y =$ (a function of x). What is the interval of existence of this solution?

Solution: Let $M = \frac{y}{x} + 6x$, $N = \ln x - 2$.

1. Verify if the equation is exact.

$$M_y = \frac{1}{x}, \quad N_x = \frac{1}{x}.$$

So the equation is exact.

2. Now find ψ such that $\psi_x = M$, $\psi_y = N$.

$$\psi_x = \frac{y}{x} + 6x \quad \Rightarrow \quad \psi = y \ln x + 3x^2 + h(y)$$
$$\psi_y = \ln x + h'(y) = \ln x - 2 \quad \Rightarrow \quad h'(y) = -2 \quad \Rightarrow \quad h(y) = -2y$$

Hence, the general solution is given implicitly by

$$\psi(x, y) = y \ln x + 3x^2 - 2y = C$$

Solving for y, we obtain

$$y = \frac{C - 3x^2}{\ln x - 2}$$

Verifying the initial condition:

$$y(1) = \frac{C - 3}{-2} = 3 \quad \Rightarrow \quad C = -3.$$

Hence,

$$y = -\frac{3 + 3x^2}{\ln x - 2}.$$

The interval of existence of this solution is the largest interval of its domain that contains $x = 1$. The function is defined for $x > 0$, $x \neq e^2$, hence the interval of existence is $(0, e^2)$.

4. Solve the initial value problem [18pt]

$$9y'' + 12y' + 8y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

Solution: 1. The characteristic equation is

$$9r^2 + 12r + 8 = 0,$$

which has roots

$$r = -\frac{2}{3} \pm i\frac{2}{3}$$

so the solution must have the form

$$y(t) = e^{-\frac{2}{3}t}(c_1 \cos \tfrac{2}{3}t + c_2 \sin \tfrac{2}{3}t).$$

2. To find c_1 and c_2 we use the initial conditions. We have

$$y'(t) = e^{\frac{2}{3}t}((-\tfrac{2}{3}c_1 + \tfrac{2}{3}c_2) \cos \tfrac{2}{3}t + (-\tfrac{2}{3}c_1 - \tfrac{2}{3}c_2) \sin \tfrac{2}{3}t).$$

So the initial conditions read

$$\begin{aligned} c_1 \qquad\quad &= 2 \\ -\tfrac{2}{3}c_1 + \tfrac{2}{3}c_2 &= -1 \end{aligned}$$

which gives $c_1 = 2$, $c_2 = \frac{1}{2}$.

Hence the solution is

$$y(t) = e^{-\frac{2}{3}t}(2 \cos \tfrac{2}{3}t + \tfrac{1}{2} \sin \tfrac{2}{3}t).$$

5. Use the method of undetermined coefficients to find the general solution of [17pt]

$$y'' + 2y' + 5y = 20\cos x.$$

Solution: 1. The characteristic equation is

$$r^2 + 2r + 5 = 0,$$

which has roots

$$r = -1 \pm 2i.$$

So the general solution is

$$y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin x) + Y(x),$$

where Y is a particular solution of the equation.

2. Since $r = i$ is not a root of the characteristic equation, we can find Y of the form

$$Y(x) = A\cos x + B\sin x.$$

We have

$$\begin{aligned} Y'(x) &= B\cos x - A\sin x \\ Y''(x) &= -A\cos x - B\sin x, \end{aligned}$$

so

$$\begin{aligned} Y'' + 2Y' + 5Y &= (4A + 2B)\cos x + (4B - 2A)\sin x \\ &= 20\cos x + 0\sin x. \end{aligned}$$

We obtain a system

$$4A + 2B = 20; \quad 4B - 2A = 0$$

which gives

$$A = 4,\ B = 2.$$

So the general solution is

$$y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin x) + 4\cos x + 2\sin x.$$

6. Find a particular solution of the equation [16pt]

$$y'' - 2y' + y = \frac{e^x}{x}$$

by the method of variation of parameters.

Solution: 1. The characteristic equation is

$$r^2 - 2r + 1 = 0$$
$$(r-1)^2 = 0.$$

So $r = 1$ is a repeated root. Hence the solutions

$$y_1(x) = e^x, \quad y_2(x) = xe^x$$

form a fundamental set of solutions.

2. The corresponding Wronskian is

$$W(x) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}.$$

3. The method of variation of parameters now gives a particular solution

$$Y = u_1 y_1 + u_2 y_2,$$

where

$$u_1(x) = -\int \frac{y_2 g}{W} dx = -\int \frac{xe^x e^x x^{-1}}{e^{2x}} dx = -\int dx = -x + C$$
$$u_2(x) = \int \frac{y_1 g}{W} dx = \int \frac{e^x e^x x^{-1}}{e^{2x}} dx = \int x^{-1} dx = \ln|x| + C$$

Taking the constants of integration zero, we obtain

$$Y = -xe^x + x\ln|x|e^x.$$

(Once can further simplify this solution by dropping the term $-xe^x$ as it is a multiple of y_2).