MA366 MIDTERM EXAM 2 – PRACTICE PROBLEM SOLUTIONS

1. The general solution of

$$y''' + 4y'' + 5y' = 0$$

is?

Solution. The characteristic equation is

$$r^3 + 4r^2 + 5 = r(r^2 + 4r + 5) = 0$$

and the roots are

$$r = 0, \quad r = -2 \pm i.$$

The corresponding solutions are

$$y_1 = 1, \quad y_2 = e^{-2t} \cos t, \quad y_3 = e^{-2t} \sin t,$$

and the general solution is given by

$$\boxed{y = C_1 + C_2 e^{-2t} \cos t + C_3 e^{-2t} \sin t.}$$

□

2. Find the solution of the initial value problem

$$y^{(4)} + 2y'' + y = 3t + 4; \quad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1$$

by using the method of undetermined coefficients.

Solution. To be added. □

3. An object weighting 8 pounds attached to a spring will stretch it 6 inches beyond its natural length. There is a damping force with a damping constant $c = 6$ lbs-sec/ft and there is no external force. If at $t = 0$ the object is pulled 2 feet below equilibrium and then released, the initial value problem describing the vertical displacement $x(t)$ becomes?

Solution. Working in lbs–ft–sec unit system. The equation is

$$mx'' + \gamma x' + kx = 0,$$

where m is the mass, $\gamma = c$ is the dumping constant (given), k is the spring stiffness. We have

$$mg = 8, \quad g = 32 \quad \Rightarrow \quad m = \frac{1}{4}.$$

To find k, we use Hooke's law

$$mg = 8 = kL = \frac{1}{2}k \quad \Rightarrow \quad k = 16.$$

So, the equation is

$$\boxed{\tfrac{1}{4} x'' + 6x' + 16x = 0}$$

and the initial condition is

$$\boxed{x(0) = 2, \quad x'(0) = 0.}$$

□

4. A spring-mass system is governed by the initial value problem

$$x'' + 4x' + 4x = 4\cos\omega t$$
$$x(0) = 9, \quad x'(0) = -2.$$

For what value(s) of ω will resonance occur?

Solution. $\boxed{\text{For no value of } \omega.}$ The resonance occurs only in undamped systems. □

5. $\mathcal{L}\{e^t(1+\cos 2t)\} = ?$

Solution.

$$\begin{aligned}
&\mathcal{L}\{e^t(1+\cos 2t)\}\\
&= \mathcal{L}\{e^t\} + \mathcal{L}\{e^t\cos 2t\}\\
&= \frac{1}{s-1} + \frac{s-1}{(s-1)^2+4}\\
&= \boxed{\frac{1}{s-1} + \frac{s-1}{s^2-2s+5}.}
\end{aligned}$$

□

6. Find the Laplace transform of

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases}.$$

Solution. Represent the function f as follows

$$\begin{aligned}
f(t) &= t - u_1(t)t\\
&= t - u_1(t)(t-1+1).
\end{aligned}$$

Then, using the formula

$$\mathcal{L}\{u_c(t)g(t-c)\} = e^{-cs}\mathcal{L}\{g\},$$

we obtain

$$\begin{aligned}
\mathcal{L}\{f\} &= \mathcal{L}\{t\} - e^{-s}\mathcal{L}\{t+1\}\\
&= \boxed{\frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right).}
\end{aligned}$$

□

7. Solve

$$y'' + 3y' + 2y = 4u_1(t)$$
$$y(0) = 0, \quad y'(0) = 1.$$

Solution. Using the Laplace transform. Let $Y = \mathcal{L}\{y\}$. Then

$$\begin{aligned}
(s^2Y - 1) + 3sY + 2Y &= \frac{4e^{-s}}{s}\\
(s^2+3s+2)Y &= 1 + \frac{4e^{-s}}{s}
\end{aligned}$$

$$\begin{aligned} Y &= \frac{1}{(s+1)(s+2)} + \frac{4e^{-s}}{s(s+1)(s+2)} \\ &= F_1(s) + e^{-s}F_2(s). \end{aligned}$$

Now find partial fraction decomposition of F_1 and F_2

$$F_1 = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\begin{aligned} F_2 &= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \\ &= \frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2}. \end{aligned}$$

This gives

$$\begin{aligned} f_1 &= \mathcal{L}^{-1}\{F_1\} = e^{-t} - e^{-2t} \\ f_2 &= \mathcal{L}^{-1}\{F_2\} = 2 - 4e^{-t} + 2e^{-2t} \end{aligned}$$

$$y = f_1(t) + u_1(t)f_2(t-1) =$$

$$\boxed{e^{-t} - e^{-2t} + u_1(t)\left(2 - 4e^{-(t-1)} + 2e^{-2(t-1)}\right)}$$

□

8. Find the solution of the initial value problem

$$\begin{aligned} &y'' + y = \delta(t-\pi) \\ &y(0) = 0, \quad y'(0) = 1. \end{aligned}$$

Solution. Using the Laplace transform. Let $Y = \mathcal{L}\{y\}$. Then we have

$$\begin{aligned} &(s^2Y - 1) + Y = e^{-\pi s} \\ &(s^2+1)Y = 1 + e^{-\pi s} \\ &Y = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} \\ &\boxed{y = \sin t + u_\pi(t)\sin(t-\pi).} \end{aligned}$$

□

9. The inverse Laplace transform of

$$F(s) = \frac{se^{-s}}{s^2+2s+5}$$

is?

Solution.

$$\begin{aligned} F(s) &= e^{-s}\frac{s}{(s+1)^2+4} \\ &= e^{-s}\left(\frac{s+1}{(s+1)^2+4} - \frac{1}{2}\frac{2}{(s+1)^2+4}\right). \end{aligned}$$

Thus,

$$\mathcal{L}^{-1}\{F\} = \boxed{u_1(t)e^{-(t-1)}(\cos 2(t-1) - \tfrac{1}{2}\sin 2(t-1))}$$

□

10. $\mathcal{L}\left\{\int_0^t \sin 2(t-\tau)\cos(3\tau)d\tau\right\} = ?$

Solution. Laplace transform of a convolution integral:

$$\begin{aligned}
&\mathcal{L}\left\{\int_0^t \sin 2(t-\tau)\cos(3\tau)d\tau\right\} \\
&= \mathcal{L}\{\sin 2t\}\mathcal{L}\{\cos 3t\} \\
&= \frac{2}{s^2+4}\cdot\frac{s}{s^2+9} \\
&= \boxed{\frac{2s}{(s^2+4)(s^2+9)}}
\end{aligned}$$

□

Answers for multiple choice problems 11-18:
11. B
12. B
13. B
14. A
15. A
16. C
17. B
18. D